

ON THE VALUE DISTRIBUTION OF THE EPSTEIN ZETA FUNCTION IN THE CRITICAL STRIP

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ABSTRACT. We study the value distribution of the Epstein zeta function $E_n(L, s)$ for $0 < s < \frac{n}{2}$ and a random lattice L of large dimension n . For any fixed $c \in (\frac{1}{4}, \frac{1}{2})$ and $n \rightarrow \infty$, we prove that the random variable $V_n^{-2c} E_n(\cdot, cn)$ has a limit distribution, which we give explicitly (here V_n is the volume of the n -dimensional unit ball). More generally, for any fixed $\varepsilon > 0$ we determine the limit distribution of the random function $c \mapsto V_n^{-2c} E_n(\cdot, cn)$, $c \in [\frac{1}{4} + \varepsilon, \frac{1}{2} - \varepsilon]$. After compensating for the pole at $c = \frac{1}{2}$ we even obtain a limit result on the whole interval $[\frac{1}{4} + \varepsilon, \frac{1}{2}]$, and as a special case we deduce the following strengthening of a result by Sarnak and Strömbergsson [15] concerning the height function $h_n(L)$ of the flat torus \mathbb{R}^n/L : The random variable $n\{h_n(L) - (\log(4\pi) - \gamma + 1)\} + \log n$ has a limit distribution as $n \rightarrow \infty$, which we give explicitly. Finally we discuss a question posed by Sarnak and Strömbergsson as to whether there exists a lattice $L \subset \mathbb{R}^n$ for which $E_n(L, s)$ has no zeros in $(0, \infty)$.

1. INTRODUCTION

Let X_n denote the space of n -dimensional lattices of covolume 1. We realize X_n as the homogeneous space $\mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R})$, where $\mathrm{SL}(n, \mathbb{Z})g$ corresponds to the lattice $\mathbb{Z}^n g \subset \mathbb{R}^n$. We further let μ_n denote the Haar measure on $\mathrm{SL}(n, \mathbb{R})$, normalized to be the unique right $\mathrm{SL}(n, \mathbb{Z})$ -invariant probability measure on X_n .

For $L \in X_n$ and $\mathrm{Re} s > \frac{n}{2}$ the Epstein zeta function is defined by

$$E_n(L, s) = \sum'_{\mathbf{m} \in L} |\mathbf{m}|^{-2s},$$

where $'$ denotes that the zero vector should be omitted. $E_n(L, s)$ has an analytic continuation to \mathbb{C} except for a simple pole at $s = \frac{n}{2}$ with residue $\pi^{\frac{n}{2}} \Gamma(\frac{n}{2})^{-1}$. Furthermore $E_n(L, s)$ satisfies the functional equation

$$(1.1) \quad F_n(L, s) = F_n(L^*, \frac{n}{2} - s),$$

where

$$(1.2) \quad F_n(L, s) := \pi^{-s} \Gamma(s) E_n(L, s),$$

and L^* is the dual lattice of L . The close relation with the Riemann zeta function, in fact $\zeta(2s) = \frac{1}{2} E_1(\mathbb{Z}, s)$, makes it natural to call the region $0 < \mathrm{Re} s < \frac{n}{2}$ the critical strip for $E_n(L, s)$. Note however that for all $n \geq 2$ there exist lattices $L \in X_n$ for which the Riemann hypothesis for $E_n(L, s)$ is known to fail (cf. [21, Thm. 1]; see also [1], [17], [20] and [22]).

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It follows from (1.1) that $E_n(L, 0) = -1$ for all $L \in X_n$. Since $E_n(L, s)$ has a simple pole at $s = \frac{n}{2}$ with positive residue it is also clear that

$$\lim_{s \rightarrow \frac{n}{2}-} E_n(L, s) = -\infty$$

for all $L \in X_n$. In this paper we will be interested in the behavior of $E_n(L, s)$ in the interval $0 < s < \frac{n}{2}$ for large n . In particular we will, for $0 < c < \frac{1}{2}$, be interested in questions concerning the value distribution of $E_n(L, cn)$ as $n \rightarrow \infty$. These questions are mainly motivated by the work of Sarnak and Strömbergsson [15] on minima of $E_n(L, s)$. They note that if there exists a lattice $L_0 \in X_n$ satisfying $E_n(L, s) \geq E_n(L_0, s)$ for all $0 < s < \frac{n}{2}$ and all $L \in X_n$ then $E_n(L_0, s) < 0$ for $0 < s < \frac{n}{2}$. Hence, for such a lattice L_0 , $E_n(L_0, s)$ has no zeros in $(0, \infty)$.

The question as to whether or not a lattice with the last property can exist is also of interest in algebraic number theory. In particular, by Hecke's integral formula (cf. [8, pp. 198-207] and [22, eq. (9)]), if we knew that $E_n(L, s) < 0$ for all $0 < s < \frac{n}{2}$ and all lattices $L \in X_n$ of a special type related to a given number field k , this would imply that the Dedekind zeta function $\zeta_k(s)$ of k satisfies $\zeta_k(s) < 0$ for all $s \in (0, 1)$!

Gaining insight into whether or not lattices $L \in X_n$ with $E_n(L, s) \neq 0$, $\forall s > 0$, do exist for all n (or all large n) is one of the main goals of the present study. A first step in this direction was taken by Sarnak and Strömbergsson in [15, Sec. 6], where they study the value distribution of the height function for flat tori as $n \rightarrow \infty$. Recall that for the flat torus \mathbb{R}^n/L , with $L \in X_n$, the height function is given by

$$(1.3) \quad h_n(\mathbb{R}^n/L) = h_n(L) = 2 \log(2\pi) + \frac{\partial}{\partial s} E_n(L^*, s)|_{s=0}.$$

Theorem 3 of [15] states that if $\varepsilon > 0$ is fixed then

$$(1.4) \quad \text{Prob}_{\mu_n} \left\{ L \in X_n \mid |h_n(L) - (\log(4\pi) - \gamma + 1)| < \varepsilon \right\} \rightarrow 1$$

as $n \rightarrow \infty$, where γ is Euler's constant. Expressed in terms of the Epstein zeta function, (1.4) says

$$(1.5) \quad \text{Prob}_{\mu_n} \left\{ L \in X_n \mid \left| \frac{\partial}{\partial s} E_n(L, s)|_{s=0} - (1 - \gamma - \log \pi) \right| < \varepsilon \right\} \rightarrow 1$$

as $n \rightarrow \infty$. Here $1 - \gamma - \log(\pi) \approx -0.72$. Note that (1.5) together with $E_n(L, 0) = -1$ ($\forall L \in X_n$) give a fairly precise description of the behavior of $E_n(L, s)$ in the left end of the interval $0 < s < \frac{n}{2}$ for most $L \in X_n$ when n is large.

The results in the present paper give information on the value distribution of $E_n(L, s)$ for $\frac{n}{4} < s < \frac{n}{2}$ with large n . Using (1.1) it is then easy to infer results also for the interval $0 < s < \frac{n}{4}$. In order to state our theorems we first need to introduce some notation. We consider a Poisson process $\mathcal{P} = \{\widehat{N}(V), V \geq 0\}$ on the positive real line with constant intensity $\frac{1}{2}$, and let T_1, T_2, T_3, \dots denote the points of the process ordered in such a way that $0 < T_1 < T_2 < T_3 < \dots$. We let $N(V) := 2\widehat{N}(V)$ and define, for all $V \geq 0$,

$$(1.6) \quad R(V) := N(V) - V.$$

Finally we let V_n denote the volume of the unit ball in \mathbb{R}^n .

Theorem 1.1. *Let $\frac{1}{4} < c_1 < c_2 < \frac{1}{2}$. For each $n \in \mathbb{Z}_{\geq 1}$ consider*

$$c \mapsto V_n^{-2c} E_n(\cdot, cn)$$

as a random function in $C([c_1, c_2])$. Then the distribution of this random function converges to the distribution of

$$c \mapsto \int_0^\infty V^{-2c} dR(V)$$

as $n \rightarrow \infty$.

For our purposes it is essential to understand $V_n^{-2c} E_n(\cdot, cn)$ as a random function. Nevertheless, for extra clarity we also state the following immediate corollary of Theorem 1.1.

Corollary 1.2. *For fixed $c \in (\frac{1}{4}, \frac{1}{2})$, the distribution of the random variable $V_n^{-2c} E_n(\cdot, cn)$ converges to the distribution of $\int_0^\infty V^{-2c} dR(V)$ as $n \rightarrow \infty$. In fact, for any $m \geq 1$ and fixed $\frac{1}{4} < c_1 < \dots < c_m < \frac{1}{2}$, the distribution of the random vector*

$$\left(V_n^{-2c_1} E_n(\cdot, c_1 n), \dots, V_n^{-2c_m} E_n(\cdot, c_m n) \right)$$

converges to the distribution of

$$\left(\int_0^\infty V^{-2c_1} dR(V), \dots, \int_0^\infty V^{-2c_m} dR(V) \right)$$

as $n \rightarrow \infty$.

The fact that the limit random variables in Theorem 1.1 and Corollary 1.2 are well-defined follows from the bound

$$(1.7) \quad |R(V)| \ll (V \log \log V)^{\frac{1}{2}} \quad \text{as } V \rightarrow \infty,$$

which holds almost surely, as a simple consequence of the law of the iterated logarithm. We also mention that the distribution of $\int_0^\infty V^{-2c} dR(V)$, for fixed $c \in (\frac{1}{4}, \frac{1}{2})$, is well understood. In particular $\int_0^\infty V^{-2c} dR(V)$ has a strictly $\frac{1}{2c}$ -stable distribution. We discuss these matters in detail in Section 2.

Let us point out the close formal similarity between the results above and our previous results in [19] on the value distribution of $E_n(\cdot, cn)$ to the right of the critical strip. In the language we have adopted here the main result in [19] states that for fixed $c > \frac{1}{2}$, the distribution of the random variable $V_n^{-2c} E_n(\cdot, cn)$ converges to the distribution of $\int_0^\infty V^{-2c} dN(V)$ as $n \rightarrow \infty$. Similar statements also hold for general finite dimensional distributions and the corresponding random functions. Hence, passing from the case to the right of the critical strip to the present one, we need only change from " $dN(V)$ " to " $dR(V)$ " in the limit variable.

A crucial ingredient in the proof of Theorem 1.1 is our result [18] on the distribution of lengths of lattice vectors in a random lattice $L \in X_n$. It says that, as $n \rightarrow \infty$, the suitably normalized non-zero vector lengths in a random lattice $L \in X_n$ behave like the points of a Poisson process on the the positive real line. To be more precise: Given a lattice $L \in X_n$, we order its non-zero vectors by increasing lengths as $\pm \mathbf{v}_1, \pm \mathbf{v}_2, \pm \mathbf{v}_3, \dots$, set $\ell_j = |\mathbf{v}_j|$ (thus $0 < \ell_1 \leq \ell_2 \leq \ell_3 \leq \dots$), and define

$$(1.8) \quad \mathcal{V}_j := \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \ell_j^n,$$

so that \mathcal{V}_j is the volume of an n -dimensional ball of radius ℓ_j . The main result in [18] now states that, as $n \rightarrow \infty$, the volumes $\{\mathcal{V}_j\}_{j=1}^\infty$ determined by a random lattice $L \in X_n$ converges in distribution to the points $\{T_j\}_{j=1}^\infty$ of the Poisson process \mathcal{P} on the positive real line with constant intensity $\frac{1}{2}$.

In view of this result from [18], the following definitions are natural. Given $L \in X_n$ and $V \geq 0$ we let $N_n(V)$ denote the number of non-zero lattice points of L in the closed n -ball of volume V centered at the origin, and define

$$(1.9) \quad R_n(V) := N_n(V) - V.$$

Note that the above-mentioned result from [18] implies in particular that $N_n(V)$ tends in distribution to $N(V)$ as $n \rightarrow \infty$, and $R_n(V)$ tends in distribution to $R(V)$, for any $V \geq 0$.

A second crucial ingredient in our proof of Theorem 1.1 is a bound of similar quality as (1.7) for the corresponding function $R_n(V)$ on X_n .

Theorem 1.3. *For all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $n \geq 3$ and $C \geq 1$ we have*

$$\text{Prob}_{\mu_n} \left\{ L \in X_n \mid |R_n(V)| \leq C_\varepsilon (CV)^{\frac{1}{2}} (\log V)^{\frac{3}{2}+\varepsilon}, \quad \forall V \geq 10 \right\} \geq 1 - C^{-1}.$$

We stress in particular that C_ε is independent of n .

Theorem 1.3 is interesting not only for being an important technical part of the proof of Theorem 1.1, but also for its connection with the famous circle problem generalized to dimension n and general ellipsoids. Given $V > 0$, $n \geq 2$ and $L \in X_n$ the problem asks for the number $\mathcal{N}(V) = 1 + N_n(V)$ of lattice points of L in the closed n -ball of volume V centered at the origin. It is well-known that $\mathcal{N}(V)$ is asymptotic to the volume V of this ball. Hence $1 + R_n(V)$ equals the remainder term in this asymptotic relation, and Theorem 1.3 implies that this remainder is $\ll V^{\frac{1}{2}} (\log V)^{\frac{3}{2}+\varepsilon}$ as $V \rightarrow \infty$, for almost every $L \in X_n$.

As far as we are aware, the fact that almost every $L \in X_n$ satisfies $|R_n(V)| \ll V^{\frac{1}{2}} (\log V)^{\frac{3}{2}+\varepsilon}$, or just $|R_n(V)| \ll V^{\frac{1}{2}+\varepsilon}$, as $V \rightarrow \infty$, has not been pointed out previously in the literature. We mention a result from 1928 by Jarník [9, Satz 3], which in our notation says that $|R_n(V)| \ll V^{\frac{1}{2}+\varepsilon}$ holds for almost every orthogonal lattice L (viz. a lattice which has an orthogonal \mathbb{Z} -basis), when $n \geq 4$. Also in this vein we mention the impressive recent work by Bentkus and Götze [2], [3] and Götze [6], which imply strong explicit bounds on $R_n(V)$ for an arbitrary given lattice L . In particular, [6] implies that $|R_n(V)| \ll V^{1-\frac{2}{n}}$ holds for every $L \in X_n$ when $n \geq 5$, and furthermore the stronger bound $R_n(V) = o(V^{1-\frac{2}{n}})$ as $V \rightarrow \infty$ whenever L is irrational in the sense that the Gram matrix for some \mathbb{Z} -basis of L (equivalently: for every \mathbb{Z} -basis of L) is not proportional to a matrix with integer entries only.

In Section 6 we extend the result in Theorem 1.1 to the case $c_2 = \frac{1}{2}$. In order for this to make sense we have to subtract the singular part of $V_n^{-2c} E_n(\cdot, cn)$ from both the random functions appearing in Theorem 1.1. A precise statement of this limit value distribution result can be found in Theorem 6.2. As an application we prove a result on the asymptotic value distribution of the height function h_n . First, in Lemma 2.9, we show that the limit

$$(1.10) \quad Z_0 := \lim_{c \rightarrow \frac{1}{2}-} \left(\int_0^\infty V^{-2c} dR(V) + \frac{1}{1-2c} \right)$$

exists almost surely. Recall that it was proved in [15, Thm. 3] that the random variable $h_n(L)$ converges in distribution to the constant $\log(4\pi) - \gamma + 1$ (cf. (1.4) above). Relating Z_0 to a similar limit involving $E_n(L, cn)$ and using the functional

equation (1.1) and the formula (1.3) for h_n , we obtain the following much more precise convergence result:

Theorem 1.4. *The random variable*

$$n \left(h_n(L) - (\log(4\pi) - \gamma + 1) \right) + \log n$$

converges in distribution to

$$2Z_0 - \log \pi - 1$$

as $n \rightarrow \infty$.

Returning to the question of whether there exists a lattice $L \in X_n$ such that $E_n(L, s) < 0$ for $0 < s < \frac{n}{2}$, we note that Theorem 1.1 and Theorem 6.2 have the following corollary.

Corollary 1.5. *For any fixed $\frac{1}{4} < c_1 < c_2 \leq \frac{1}{2}$, the limit*

$$\lim_{n \rightarrow \infty} \text{Prob}_{\mu_n} \left\{ L \in X_n \mid E_n(L, s) < 0 \text{ for all } s \in [c_1 n, c_2 n] \setminus \{\frac{1}{2}n\} \right\}$$

exists, and equals

$$f(c_1, c_2) := \text{Prob} \left\{ \int_0^\infty V^{-2c} dR(V) < 0 \text{ for all } c \in [c_1, c_2] \setminus \{\frac{1}{2}\} \right\}.$$

Moreover, for all $\frac{1}{4} < c_1 < c_2 \leq \frac{1}{2}$ the probability $f(c_1, c_2)$ satisfies $0 < f(c_1, c_2) < 1$.

In particular, for any given $\varepsilon > 0$ the probability that

$$E_n(L, s) < 0 \quad \text{for all } s \in \left[\left(\frac{1}{4} + \varepsilon \right) n, \frac{1}{2} n \right]$$

holds tends to a positive limit as $n \rightarrow \infty$! However, we also have the following results.

Theorem 1.6. *Fix $m \in \mathbb{Z}_{\geq 1}$ and let $c_j = \frac{1}{4} + \eta_j$ with $\eta_j \in (0, \frac{1}{4})$ for $1 \leq j \leq m$. If (η_1, \dots, η_m) tends to the zero vector in \mathbb{R}^m in such a way that $\eta_j / \eta_{j+1} \rightarrow 0$ for each $1 \leq j \leq m-1$, then the m -dimensional random vector*

$$\left(\left(2c_1 - \frac{1}{2} \right)^{\frac{1}{2}} \int_0^\infty V^{-2c_1} dR(V), \dots, \left(2c_m - \frac{1}{2} \right)^{\frac{1}{2}} \int_0^\infty V^{-2c_m} dR(V) \right)$$

converges in distribution to the distribution of m independent $N(0, 1)$ -variables.

Corollary 1.7. *For each fixed $c_2 \in (\frac{1}{4}, \frac{1}{2}]$, the probability $f(c_1, c_2)$ tends to zero as $c_1 \rightarrow \frac{1}{4}+$.*

As an immediate consequence it follows that for any $\varepsilon > 0$ we have

$$\text{Prob}_{\mu_n} \left\{ L \in X_n \mid E_n(L, s) < 0 \text{ for all } s \in \left[\frac{1}{4} n, \left(\frac{1}{4} + \varepsilon \right) n \right] \right\} \rightarrow 0$$

as $n \rightarrow \infty$. In particular this entails that the probability that $E_n(L, s)$ has a zero in $(0, \infty)$ tends to one as $n \rightarrow \infty$. Hence the question of Sarnak and Strömbergsson is rather delicate!

Finally we remark that the precise behavior of the random variable $E_n(L, cn)$ for $c = \frac{1}{4}$ or c tending to $\frac{1}{4}$ as $n \rightarrow \infty$ remains very much an open and exciting question, which we hope to tackle in future work.

2. THE RANDOM VARIABLES $H(c)$ AND Z_0

2.1. The random variable $H(c)$. In this section we prove some basic results about the random variable

$$(2.1) \quad H(c) := \int_0^\infty V^{-2c} dR(V),$$

which appears as the limit variable in Theorem 1.1 and Corollary 1.2.

Recall from the introduction that, for a Poisson process $\mathcal{P} = \{\hat{N}(V), V \geq 0\}$ on the positive real line with constant intensity $\frac{1}{2}$, we let $N(V) := 2\hat{N}(V)$ and define

$$R(V) := N(V) - V, \quad V \geq 0.$$

We also recall that $\hat{N}(V)$ denotes the number of points of \mathcal{P} falling in the interval $(0, V]$ and that $\hat{N}(V)$ is Poisson distributed with expectation value $\frac{1}{2}V$. In fact, since furthermore $\hat{N}(V_2) - \hat{N}(V_1)$ is Poisson distributed with expectation value $\frac{1}{2}(V_2 - V_1)$, it follows that $\mathbb{E}(R(V_2) - R(V_1)) = 0$ and

$$(2.2) \quad \mathbb{E}\left((R(V_2) - R(V_1))^2\right) = \text{Var}(R(V_2) - R(V_1)) = \text{Var}(N(V_2) - N(V_1)) = 2(V_2 - V_1)$$

for all $0 \leq V_1 < V_2$. We let T_1, T_2, T_3, \dots denote the points of \mathcal{P} ordered in such a way that $0 < T_1 < T_2 < T_3 < \dots$. Hence the sequence $\{T_j\}_{j=1}^\infty$ belongs to the space

$$\Omega := \left\{ \mathbf{x} = \{x_j\}_{j=1}^\infty \in (\mathbb{R}_{\geq 0})^\infty \mid 0 < x_1 < x_2 < x_3 < \dots \right\}.$$

We equip Ω with the subspace topology induced from the product topology on $(\mathbb{R}_{\geq 0})^\infty$. We denote the distribution of \mathcal{P} on Ω by \mathbf{P} and note that \mathbf{P} is actually a Borel probability measure on Ω .

To begin with we need an estimate of $R(V)$. Using the law of the iterated logarithm (see [7]) it is straightforward to show that with probability one we have

$$\limsup_{V \rightarrow \infty} \frac{|R(V)|}{(V \log \log V)^{\frac{1}{2}}} = 2.$$

In particular it follows that with probability one there exists a constant $C > 2$ (that depends on $\mathbf{x} \in \Omega$) such that

$$(2.3) \quad |R(V)| < C(V \log \log V)^{\frac{1}{2}}, \quad \forall V \geq 10.$$

In the following lemma we give a simple proof of a slightly weaker bound than (2.3), which as input only uses the monotonicity of $N(V)$ and the variance relation (2.2). This proof has the advantage that it easily generalizes to the situation in Theorem 1.3 (see Section 3).

Lemma 2.1. *For all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $C \geq 1$ we have*

$$\mathbf{P} \left\{ |R(V)| \leq C_\varepsilon (CV)^{\frac{1}{2}} (\log V)^{\frac{3}{2} + \varepsilon}, \quad \forall V \geq 10 \right\} \geq 1 - C^{-1}.$$

Remark 2.2. Note that the set

$$\left\{ \mathbf{x} \in \Omega \mid |R(V)| \leq C_\varepsilon (CV)^{\frac{1}{2}} (\log V)^{\frac{3}{2} + \varepsilon}, \quad \forall V \geq 10 \right\}$$

is indeed \mathbf{P} -measurable, viz. a Borel subset of Ω . Indeed, since $R(V)$ is right-continuous for every $\mathbf{x} \in \Omega$, the above set equals the countable intersection

$$\bigcap_{V \in \mathbb{Q} \cap [10, \infty)} \left\{ \mathbf{x} \in \Omega \mid |R(V)| \leq C_\varepsilon (CV)^{\frac{1}{2}} (\log V)^{\frac{3}{2} + \varepsilon} \right\}.$$

Here each set is of the form $\{\mathbf{x} \in \Omega \mid |R(V)| \leq A\}$ for some $V, A \geq 0$, and since

$$\{\mathbf{x} \in \Omega \mid |R(V)| \leq A\} = \{\mathbf{x} \in \Omega \mid x_m \leq V \text{ and } x_{\ell+1} > V\}$$

with $m = \lceil \frac{1}{2}(V - A) \rceil$ and $\ell = \lfloor \frac{1}{2}(V + A) \rfloor$, this is a Borel subset of Ω . In a similar way one also proves that the set Ω_ε defined below in (2.11) is a Borel subset of Ω , and also that the set considered in Theorem 1.3 is a Borel subset of X_n .

Proof of Lemma 2.1. For all $A \geq 10$, it follows from (2.2) that

$$\begin{aligned} \mathbb{E} \left(R(A)^2 + \sum_{0 \leq k \leq \frac{1}{2} \log_2 A} \sum_{j=0}^{2^k-1} \left(R((1 + 2^{-k}(j+1))A) - R((1 + 2^{-k}j)A) \right)^2 \right) \\ = 2A + \sum_{0 \leq k \leq \frac{1}{2} \log_2 A} 2^k \cdot 2 \cdot 2^{-k} A = 2A (\lfloor \tfrac{1}{2} \log_2 A \rfloor + 2) \ll A \log A, \end{aligned}$$

where the implied constant is absolute. Hence, using Markov's inequality, we get

$$(2.4) \quad \mathbf{P} \left\{ R(A)^2 + \sum_{0 \leq k \leq \frac{1}{2} \log_2 A} \sum_{j=0}^{2^k-1} \left(R((1 + 2^{-k}(j+1))A) - R((1 + 2^{-k}j)A) \right)^2 \geq CA \log A \right\} \ll C^{-1},$$

uniformly over all $C > 0$ and $A \geq 10$. On the other hand we claim that for all $C \geq 1$, $A \geq 10$ and $\mathbf{x} \in \Omega$ for which

$$(2.5) \quad R(A)^2 + \sum_{0 \leq k \leq \frac{1}{2} \log_2 A} \sum_{j=0}^{2^k-1} \left(R((1 + 2^{-k}(j+1))A) - R((1 + 2^{-k}j)A) \right)^2 < CA \log A$$

holds, we have

$$(2.6) \quad |R(V)| \ll (CA)^{\frac{1}{2}} \log A \quad \text{for all } V \in [A, 2A],$$

with an absolute implied constant.

To prove the claim we fix any $V \in [A, 2A]$. We also set $k_0 := \lfloor \frac{1}{2} \log_2 A \rfloor$ and let m be the largest integer satisfying $(1 + 2^{-k_0}m)A \leq V$; thus $0 \leq m \leq 2^{k_0}$. By considering the binary representation of m , we may express

$$R((1 + 2^{-k_0}m)A) - R(A)$$

as a sum of terms of the form

$$R((1 + 2^{-k}(j+1))A) - R((1 + 2^{-k}j)A),$$

where $0 \leq k \leq k_0$ and where for each $k \in \{0, \dots, k_0\}$, we either have no term, or exactly one term, for some $j = j(k, m) \in \{0, \dots, 2^k - 1\}$. Hence the total number

of terms does not exceed $k_0 + 1$ and by the Cauchy-Schwarz inequality and (2.5) we have

$$\begin{aligned} & |R((1 + 2^{-k_0}m)A) - R(A)| \\ & \leq (k_0 + 1)^{\frac{1}{2}} \left(\sum_{0 \leq k \leq \frac{1}{2} \log_2 A} \sum_{j=0}^{2^k-1} \left(R((1 + 2^{-k}(j+1))A) - R((1 + 2^{-k}j)A) \right)^2 \right)^{\frac{1}{2}} \\ & < (k_0 + 1)^{\frac{1}{2}} (CA \log A)^{\frac{1}{2}} \ll (CA)^{\frac{1}{2}} \log A. \end{aligned}$$

Using (2.5) once more we get $R(A) < (CA \log A)^{\frac{1}{2}}$ and thus, by the triangle inequality, we conclude that

$$(2.7) \quad |R((1 + 2^{-k_0}m)A)| \ll (CA)^{\frac{1}{2}} \log A.$$

Now, if $V = 2A$ then $m = 2^{k_0}$, $R(V) = R((1 + 2^{-k_0}m)A)$ and (2.7) is the desired estimate. Next we assume that $V < 2A$. Then $m + 1 \leq 2^{k_0}$ and by the argument proving (2.7) we also get

$$(2.8) \quad |R((1 + 2^{-k_0}(m+1))A)| \ll (CA)^{\frac{1}{2}} \log A.$$

Using the definition of $R(X)$ and the fact that $N(X)$ is an increasing function of X we obtain

$$R(X) \leq R(X') + X' - X \quad \text{for all } 0 \leq X \leq X'.$$

Thus, since we by our choice of m have

$$(1 + 2^{-k_0}m)A \leq V < (1 + 2^{-k_0}(m+1))A,$$

we get

$$(2.9) \quad R((1 + 2^{-k_0}m)A) - 2^{-k_0}A \leq R(V) \leq R((1 + 2^{-k_0}(m+1))A) + 2^{-k_0}A.$$

Recalling that $k_0 > \frac{1}{2} \log_2 A - 1$ we obtain $2^{-k_0}A < 2A^{\frac{1}{2}}$. Hence (2.7), (2.8) and (2.9) together conclude the proof of the claim that (2.5) implies (2.6).

Combining (2.4) with the fact that (2.5) implies (2.6), yields the following statement: There exists an absolute constant $C_0 > 0$ such that for all $C \geq 1$ and $A \geq 10$ we have

$$(2.10) \quad \mathbf{P} \left\{ \exists V \in [A, 2A] : |R(V)| > C_0 (CA)^{\frac{1}{2}} \log A \right\} \leq C^{-1}.$$

(Note that the constant C in (2.10) is an appropriate multiple of the constant C in (2.4)-(2.6).) Now, given $K \geq 1$ and $\varepsilon > 0$, we apply, for all $j \in \mathbb{Z}_{\geq 1}$, (2.10) with $A = 5 \cdot 2^j$ and $C = Kj^{1+\varepsilon}$. We conclude that there exists a constant $C'_0 > 0$, which only depends on ε , such that

$$\mathbf{P} \left\{ \exists V \in [5 \cdot 2^j, 10 \cdot 2^j] : |R(V)| > C'_0 (KV)^{\frac{1}{2}} (\log V)^{\frac{3}{2} + \frac{1}{2}\varepsilon} \right\} \leq K^{-1} j^{-1-\varepsilon}$$

for all $j \in \mathbb{Z}_{\geq 1}$. Hence, using the subadditivity of \mathbf{P} , we obtain

$$\mathbf{P} \left\{ \exists V \geq 10 : |R(V)| > C'_0 (KV)^{\frac{1}{2}} (\log V)^{\frac{3}{2} + \frac{1}{2}\varepsilon} \right\} \leq K^{-1} C',$$

where $C' := \sum_{j=1}^{\infty} j^{-1-\varepsilon} > 1$ only depends on ε . Finally, the lemma follows from setting $C = KC'^{-1}$ and $C_{\varepsilon/2} = C'_0 C'^{\frac{1}{2}}$. \square

For $\varepsilon > 0$ we define

$$(2.11) \quad \Omega_\varepsilon := \left\{ \mathbf{x} \in \Omega \mid |R(V)| \ll_{\mathbf{x}, \varepsilon} V^{\frac{1}{2}} (\log V)^{\frac{3}{2} + \varepsilon} \quad \forall V \geq 10 \right\}.$$

Note that it follows from Lemma 2.1 that $\mathbf{P}(\Omega_\varepsilon) = 1$ for every $\varepsilon > 0$. For notational convenience we will only work with $\Omega_{1/2}$ in the following; however any other set Ω_ε would do just as well. The following lemma shows that the integral $H(c)$ in (2.1) converges almost surely.

Lemma 2.3. *For every $\mathbf{x} \in \Omega_{1/2}$ the integral $H(c)$ converges for all $c \in (\frac{1}{4}, \frac{1}{2})$, and furthermore the integral $\int_A^\infty V^{-2c} dR(V)$ converges for all $A > 0$ and $c > \frac{1}{4}$.*

Proof. Let $\mathbf{x} \in \Omega_{1/2}$ be fixed. Now for any $0 < A < B$ and $c > \frac{1}{4}$ we have

$$(2.12) \quad \int_A^B V^{-2c} dR(V) = \left[V^{-2c} R(V) \right]_{V=A}^{V=B} + 2c \int_A^B V^{-2c-1} R(V) dV,$$

and since $V^{-2c} |R(V)| \ll_{\mathbf{x}} V^{\frac{1}{2}-2c} (\log V)^2$ as $V \rightarrow \infty$, with $\frac{1}{2} - 2c < 0$, it follows that both terms in the right hand side of (2.12) are convergent as $B \rightarrow \infty$. This proves the second statement of the lemma. Finally, since $R(V) = -V$ for all $0 \leq V \ll_{\mathbf{x}} 1$ it follows that if $c < \frac{1}{2}$ then the two terms in the right hand side of (2.12) are also convergent as $A \rightarrow 0$, so that $H(c)$ converges for all $c \in (\frac{1}{4}, \frac{1}{2})$. \square

Lemma 2.4. *$H(c)$ is a well-defined random variable on $\Omega_{1/2}$ for all $c \in (\frac{1}{4}, \frac{1}{2})$.*

Proof. Fix $c \in (\frac{1}{4}, \frac{1}{2})$. By Lemma 2.3, $H(c)$ is convergent for each $\mathbf{x} \in \Omega_{1/2}$, and it remains to show that $\mathbf{x} \mapsto H(c)$ is measurable. Let us for $A > 0$ consider the function $f_A : \Omega \rightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$(2.13) \quad \begin{aligned} f_A(T_1, T_2, \dots) &= \int_0^A V^{-2c} dR(V) = \int_0^A V^{-2c} dN(V) - \int_0^A V^{-2c} dV \\ &= 2 \sum_{T_j \leq A} T_j^{-2c} - \frac{A^{1-2c}}{1-2c}. \end{aligned}$$

We express Ω as a disjoint union of Borel sets as follows: $\Omega = (\cup_{j=0}^\infty \Omega^{(j)}) \cup \Omega^{(\infty)}$, where

$$(2.14) \quad \Omega^{(\infty)} = \{\mathbf{x} \in \Omega \mid x_\ell \leq A, \forall \ell\}, \quad \Omega^{(0)} = \{\mathbf{x} \in \Omega \mid A < x_1\},$$

and

$$(2.15) \quad \Omega^{(j)} = \{\mathbf{x} \in \Omega \mid x_j \leq A < x_{j+1}\} \text{ for } j \geq 1.$$

It follows from the last expression in (2.13) that the restriction of f_A to each set $\Omega^{(j)}$ is continuous (we set $f_A := \infty$ for all $\mathbf{x} \in \Omega^{(\infty)}$). Hence each f_A is measurable, and hence also the restrictions of these functions to $\Omega_{1/2}$ are measurable (of course we also have $\Omega^{(\infty)} \cap \Omega_{1/2} = \emptyset$, so that f_A is real-valued on $\Omega_{1/2}$). Thus also $H(c)$ is measurable on $\Omega_{1/2}$, since it is the pointwise limit of the sequence f_1, f_2, f_3, \dots of measurable functions. \square

Remark 2.5. We want to consider $H(c)$ also as a random variable on Ω . To make this rigorous we should redefine $H(c)$ (as for example zero) on $\Omega \setminus \Omega_{1/2}$ in order to make $H(c)$ measurable on Ω (cf. [13, p. 29]). However, since we in the present paper are only interested in questions of distribution and $\Omega_{1/2}$ has full measure in Ω , we will simply let $H(c)$ remain undefined at points where the integral is divergent.

We next note that Lemma 2.1 also implies that the tail of $H(c)$ can be made uniformly small in closed intervals $[c_1, c_2] \subset (\frac{1}{4}, \frac{1}{2}]$.

Lemma 2.6. *Let $\frac{1}{4} < c_1 < c_2 \leq \frac{1}{2}$. Then for all $\varepsilon' > 0$ there exists a constant $A_0 > 0$ such that for all $A \geq A_0$ we have*

$$\mathbf{P} \left\{ \sup_{c \in [c_1, c_2]} \left| \int_A^\infty V^{-2c} dR(V) \right| \leq \varepsilon' \right\} \geq 1 - \varepsilon'.$$

Proof. Let $\varepsilon' > 0$ and $\delta \in (0, 2c_1 - \frac{1}{2})$ be given. It follows from Lemma 2.1 that there exists a set $\Omega' \subset \Omega_{1/2}$ with $\mathbf{P}(\Omega') \geq 1 - \varepsilon'$ such that for all $\mathbf{x} \in \Omega'$ and all $V \geq 10$ we have $|R(V)| \ll_{\varepsilon', \delta} V^{\frac{1}{2} + \delta}$, where the implied constant is independent of \mathbf{x} . Now, for any $\mathbf{x} \in \Omega'$ and all $A \geq 10$, we have

$$(2.16) \quad \left| \int_A^\infty V^{-2c} dR(V) \right| = \left| \left[V^{-2c} R(V) \right]_{V=A}^{V=\infty} + 2c \int_A^\infty V^{-2c-1} R(V) dV \right| \\ \ll_{\varepsilon', \delta, c_1} A^{-2c + \frac{1}{2} + \delta} \leq A^{-2c_1 + \frac{1}{2} + \delta},$$

uniformly over all $c \in [c_1, c_2]$. Since we can make the right hand side in (2.16) as small as we like, by choosing A large enough, the lemma follows. \square

Lemma 2.7. *Let $\frac{1}{4} < c_1 < c_2 < \frac{1}{2}$. Then, for all $\mathbf{x} \in \Omega_{1/2}$ the function $c \mapsto H(c)$ is continuous in $[c_1, \frac{1}{2})$. In particular $\mathcal{H} : \Omega_{1/2} \rightarrow C([c_1, c_2])$ given by $\mathbf{x} \mapsto (c \mapsto H(c))$ is a well-defined random function.*

Proof. Fix $\mathbf{x} \in \Omega_{1/2}$. For each $A > 0$, the formula (2.13) shows that $c \mapsto \int_0^A V^{-2c} dR(V)$ is a continuous function on $[c_1, \frac{1}{2})$. Furthermore, by mimicking the proof of Lemma 2.6 we see that the function $c \mapsto H(c)$ is the uniform limit of $c \mapsto \int_0^A V^{-2c} dR(V)$ as $A \rightarrow \infty$. Hence $c \mapsto H(c)$ is indeed continuous in $[c_1, \frac{1}{2})$. The second statement now follows from Lemma 2.4 (cf., e.g., [5, p. 84]). \square

Remark 2.8. We will also consider \mathcal{H} as a random function on Ω (cf. Remark 2.5).

2.2. The random variable Z_0 . We now show that the random variable Z_0 , introduced in (1.10), is well-defined.

Lemma 2.9. *For every $\mathbf{x} \in \Omega_{1/2}$ the limit*

$$Z_0 := \lim_{c \rightarrow \frac{1}{2}^-} \left(\int_0^\infty V^{-2c} dR(V) + \frac{1}{1-2c} \right)$$

exists. In particular, this limit exists \mathbf{P} almost surely.

Proof. For any $\mathbf{x} \in \Omega_{1/2}$, $A > 0$ and $c \in (\frac{1}{4}, \frac{1}{2})$ we have

$$(2.17) \quad \int_0^\infty V^{-2c} dR(V) + \frac{1}{1-2c} \\ = \int_0^A V^{-2c} dN(V) - \int_0^A V^{-2c} dV + \int_A^\infty V^{-2c} dR(V) + \frac{1}{1-2c} \\ = \int_0^A V^{-2c} dN(V) + \left(-A^{-2c} R(A) + 2c \int_A^\infty V^{-2c-1} R(V) dV \right) + \frac{1 - A^{1-2c}}{1-2c}.$$

We recall that $\int_0^A V^{-2c} dN(V) = 2 \sum_{T_j \leq A} T_j^{-2c}$ is a finite sum and note that the integral $\int_A^\infty V^{-2c-1} R(V) dV$ is absolutely convergent. Hence, for any x and A as above, we can let $c \rightarrow \frac{1}{2}$ in the last line of (2.17) to obtain

$$\begin{aligned}
 (2.18) \quad & \lim_{c \rightarrow \frac{1}{2}-} \left(\int_0^\infty V^{-2c} dR(V) + \frac{1}{1-2c} \right) \\
 &= \int_0^A V^{-1} dN(V) + \left(-A^{-1}R(A) + \int_A^\infty V^{-2} R(V) dV \right) - \left(\frac{d}{dt} A^t \right)_{|t=0} \\
 &= \int_0^A V^{-1} dN(V) + \int_A^\infty V^{-1} dR(V) - \log A.
 \end{aligned}$$

Since $\mathbf{P}(\Omega_{1/2}) = 1$ the proof is complete. \square

Remark 2.10. Since the restriction of Z_0 to $\Omega_{1/2}$ is (by definition) a pointwise limit of measurable functions, we find that Z_0 is a random variable on $\Omega_{1/2}$. In fact we will consider Z_0 also as a random variable on Ω (cf. Remark 2.5).

Remark 2.11. We note that the last line of (2.18) gives a formula for Z_0 for any $A > 0$. In particular we have

$$Z_0 = \int_0^1 V^{-1} dN(V) + \int_1^\infty V^{-1} dR(V).$$

2.3. $H(c)$ and Z_0 have stable distributions. Even though the random variable $H(c)$ has a rather complicated definition, its distribution can be understood in very explicit terms. More precisely it follows from [14, Thm. 1.4.5] (slightly modified to allow for the Poisson process to have intensity $\frac{1}{2}$) that $H(c)$ has the strictly $\frac{1}{2c}$ -stable distribution

$$(2.19) \quad S_{\frac{1}{2c}} \left(2 \left(\frac{\Gamma(2 - \frac{1}{2c}) \cos(\frac{\pi}{4c})}{2(1 - \frac{1}{2c})} \right)^{2c}, 1, 0 \right).$$

(Here we use the same parameterization of stable distributions as [14].)

Remark 2.12. Recalling from the introduction the relation between $H(c)$ and the random variable $\int_0^\infty V^{-2c} dN(V)$, defined for $c > \frac{1}{2}$, it is interesting to note that also $\int_0^\infty V^{-2c} dN(V)$ has a strictly $\frac{1}{2c}$ -stable distribution given by the expression (2.19) (cf. [19, Sec. 2.5]).

Remark 2.13. It follows from (2.19) and [14, Property 1.2.3] that, for any $c \in (\frac{1}{4}, \frac{1}{2})$, the random variable $(2c - \frac{1}{2})^{\frac{1}{2}} H(c)$ has the strictly $\frac{1}{2c}$ -stable distribution

$$S_{\frac{1}{2c}} \left(2(2c - \frac{1}{2})^{\frac{1}{2}} \left(\frac{\Gamma(2 - \frac{1}{2c}) \cos(\frac{\pi}{4c})}{2(1 - \frac{1}{2c})} \right)^{2c}, 1, 0 \right).$$

Hence, since

$$\lim_{c \rightarrow \frac{1}{4}+} \left(\frac{1}{2c}, 2(2c - \frac{1}{2})^{\frac{1}{2}} \left(\frac{\Gamma(2 - \frac{1}{2c}) \cos(\frac{\pi}{4c})}{2(1 - \frac{1}{2c})} \right)^{2c}, 1, 0 \right) = (2, \frac{1}{\sqrt{2}}, 1, 0)$$

and $S_2(\frac{1}{\sqrt{2}}, 1, 0) = S_2(\frac{1}{\sqrt{2}}, 0, 0) = N(0, 1)$, we conclude, using [14, Def. 1.1.6] and [4, Thm. 26.3], that $(2c - \frac{1}{2})^{\frac{1}{2}} H(c)$ converges in distribution to $N(0, 1)$ as $c \rightarrow \frac{1}{4}+$. Note in particular that this proves Theorem 1.6 in the case $m = 1$.

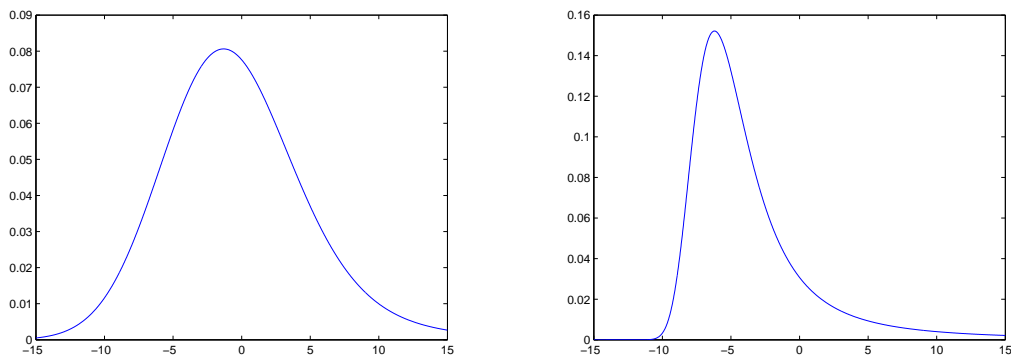


FIGURE 1. The probability density functions of $H(\frac{5}{18})$ (left) and $H(\frac{5}{12})$ (right). The figures were generated by the program STABLE, which is available from J. P. Nolan's website <http://academic2.american.edu/~jpnolan/>.

By an argument similar to the one in Remark 2.13 we now show that also Z_0 has a stable distribution. First we define, for each $c \in (\frac{1}{4}, \frac{1}{2})$, the random variable

$$\hat{H}(c) := H(c) + \frac{1}{1-2c},$$

so that $\hat{H}(c)$ tends in distribution to Z_0 as $c \rightarrow \frac{1}{2}-$. It follows from (2.19) and [14, Property 1.2.2] that $\hat{H}(c)$ has the stable distribution $S_{\alpha(c)}(\sigma(c), \beta(c), \mu(c))$, where

$$(2.20) \quad (\alpha(c), \sigma(c), \beta(c), \mu(c)) = \left(\frac{1}{2c}, 2 \left(\frac{\Gamma(2 - \frac{1}{2c}) \cos(\frac{\pi}{4c})}{2(1 - \frac{1}{2c})} \right)^{2c}, 1, \frac{1}{1-2c} \right).$$

Note that, since $\lim_{c \rightarrow \frac{1}{2}-} \alpha(c) = 1$ and the characteristic function for a stable distribution (in this parameterization) does not vary continuously with respect to α at $\alpha = 1$, we cannot take the limit directly in (2.20). However, using [14, p. 7, Rem. 4], we find that $\hat{H}(c)$ tends in distribution to $S_\alpha(\sigma, \beta, \mu)$, where

$$\begin{aligned} (\alpha, \sigma, \beta, \mu) &= \lim_{c \rightarrow \frac{1}{2}-} \left(\alpha(c), \sigma(c), \beta(c), \mu(c) + \beta(c)\sigma(c)^{\alpha(c)} \tan\left(\frac{\pi\alpha(c)}{2}\right) \right) \\ &= \left(1, \frac{\pi}{2}, 1, 1 - \log 2 - \gamma \right). \end{aligned}$$

(Here γ is Euler's constant.) Hence we conclude that Z_0 has the 1-stable distribution $S_1(\frac{\pi}{2}, 1, 1 - \log 2 - \gamma)$.

3. PROOF OF THEOREM 1.3

Recall that the proof of Lemma 2.1 only uses the monotonicity of $N(V)$ and the variance relation (2.2) (where an upper bound " $\ll V_2 - V_1$ " suffices), and makes no further use of the fact that $R(V)$ is defined in terms of a Poisson process. For this reason, it turns out that the proof of Theorem 1.3 can be completed by a direct mimic of the proof of Lemma 2.1, once we have Lemma 3.1 below.

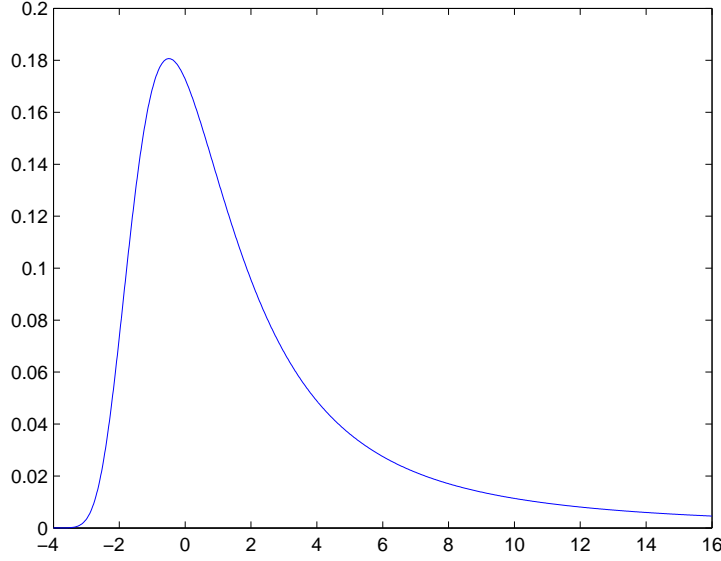


FIGURE 2. The probability density function of Z_0 . The figure was generated by the program STABLE, which is available from J. P. Nolan's website <http://academic2.american.edu/~jpnolan/>.

Lemma 3.1. *For all $A \geq 0$, $\Delta > 0$ and $n \geq 3$ we have*

$$(3.1) \quad \mathbb{E}\left(\left(R_n(A + \Delta) - R_n(A)\right)^2\right) < 5\Delta.$$

Note that it follows from Siegel's mean value formula [16] that

$$\mathbb{E}(R_n(A + \Delta) - R_n(A)) = 0,$$

and hence also that the left hand side of (3.1) equals the variance of $R_n(A + \Delta) - R_n(A)$.

Proof of Lemma 3.1. Recall that V_n denotes the volume of the unit ball in \mathbb{R}^n and that $V_n = \omega_n/n$, where ω_n is the $(n - 1)$ -dimensional volume of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. To begin with we note that

$$\begin{aligned} \mathbb{E}\left(\left(R_n(A + \Delta) - R_n(A)\right)^2\right) &= \mathbb{E}\left(\left(N_n(A + \Delta) - N_n(A)\right)^2\right) - \Delta^2 \\ &= \int_{X_n} \sum_{\mathbf{m}_1, \mathbf{m}_2 \in L} I\left(V_n |\mathbf{m}_1|^n, V_n |\mathbf{m}_2|^n \in (A, A + \Delta]\right) d\mu_n(L) - \Delta^2. \end{aligned}$$

Now recall that for any nonnegative Borel measurable function ρ on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $\rho(\pm \mathbf{x}_1, \pm \mathbf{x}_2) = \rho(\mathbf{x}_1, \mathbf{x}_2)$, Rogers' mean value formula states that (cf. [11, Thm. 4])

$$\begin{aligned}
 & \int_{X_n} \sum_{\mathbf{m}_1, \mathbf{m}_2 \in L \setminus \{\mathbf{0}\}} \rho(\mathbf{m}_1, \mathbf{m}_2) d\mu_n(L) \\
 (3.2) \quad &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2 + \sum_{e_1=1}^{\infty} \sum_{\substack{e_2 \in \mathbb{Z} \setminus \{0\} \\ \gcd(e_1, e_2)=1}} \frac{1}{e_1^n} \int_{\mathbb{R}^n} \rho\left(\mathbf{x}, \frac{e_2}{e_1} \mathbf{x}\right) d\mathbf{x} \\
 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2 + \frac{2}{\zeta(n)} \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{\infty} \int_{\mathbb{R}^n} \rho(d_1 \mathbf{x}, d_2 \mathbf{x}) d\mathbf{x}.
 \end{aligned}$$

Applying (3.2) with the function

$$\rho(\mathbf{x}_1, \mathbf{x}_2) := I\left(|\mathbf{x}_1|, |\mathbf{x}_2| \in \left(V_n^{-\frac{1}{n}} A^{\frac{1}{n}}, V_n^{-\frac{1}{n}} (A + \Delta)^{\frac{1}{n}}\right]\right)$$

yields

$$\begin{aligned}
 \mathbb{E}\left((R_n(A + \Delta) - R_n(A))^2\right) &= \frac{2}{\zeta(n)} \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{\infty} \int_{\mathbb{R}^n} \rho(d_1 \mathbf{x}, d_2 \mathbf{x}) d\mathbf{x} \\
 &= \frac{2}{\zeta(n)} \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{\infty} \omega_n \int_0^{\infty} I\left(d_1 r, d_2 r \in \left(V_n^{-\frac{1}{n}} A^{\frac{1}{n}}, V_n^{-\frac{1}{n}} (A + \Delta)^{\frac{1}{n}}\right]\right) r^{n-1} dr \\
 &= \frac{2\omega_n}{\zeta(n)} \sum_{1 \leq d_1 \leq d_2} \frac{2 - I(d_1 = d_2)}{d_1^n} \int_0^{\infty} I\left(u, \frac{d_2}{d_1} u \in \left(V_n^{-\frac{1}{n}} A^{\frac{1}{n}}, V_n^{-\frac{1}{n}} (A + \Delta)^{\frac{1}{n}}\right]\right) u^{n-1} du \\
 &= \frac{2\omega_n}{\zeta(n)} \sum_{1 \leq d_1 \leq d_2 < (1+\Delta/A)^{1/n} d_1} \frac{2 - I(d_1 = d_2)}{d_1^n} \left[\frac{u^n}{n}\right]_{u=V_n^{-\frac{1}{n}} A^{\frac{1}{n}}}^{u=V_n^{-\frac{1}{n}} (d_1/d_2)(A+\Delta)^{\frac{1}{n}}} \\
 &= \frac{2}{\zeta(n)} \sum_{1 \leq d_1 \leq d_2 < (1+\Delta/A)^{1/n} d_1} \frac{2 - I(d_1 = d_2)}{d_1^n} \left(\left(\frac{d_1}{d_2}\right)^n (A + \Delta) - A\right) \\
 &= 2\Delta + \frac{4}{\zeta(n)} \sum_{1 \leq d_1 < d_2 < (1+\Delta/A)^{1/n} d_1} d_1^{-n} \left(\left(\frac{d_1}{d_2}\right)^n (A + \Delta) - A\right).
 \end{aligned}$$

Note that for $1 \leq d_1 < d_2$ we have $\left(\frac{d_1}{d_2}\right)^n (A + \Delta) - A < \left(\frac{d_1}{d_2}\right)^n \Delta$. Hence

$$\begin{aligned}
 \mathbb{E}\left((R_n(A + \Delta) - R_n(A))^2\right) &< 2\Delta + \frac{4}{\zeta(n)} \sum_{d_1=1}^{\infty} \sum_{d_2=d_1+1}^{\infty} d_2^{-n} \Delta \\
 &< 2\Delta + \frac{4}{\zeta(n)} \sum_{d_1=1}^{\infty} \left(\int_{d_1}^{\infty} x^{-n} dx\right) \Delta \\
 &= \left(2 + \frac{4\zeta(n-1)}{(n-1)\zeta(n)}\right) \Delta < 5\Delta,
 \end{aligned}$$

which is the desired bound. \square

4. TREATMENT OF THE EPSTEIN ZETA FUNCTION

When working with the Epstein zeta function in the critical strip it is often convenient to consider the normalized function $F_n(L, s)$ (cf. (1.2)). In particular this function has a simple expansion into incomplete gamma functions (cf. [22, Thm. 2]);

$$(4.1) \quad F_n(L, s) = \left(-\frac{1}{\frac{n}{2} - s} + \sum'_{\mathbf{m} \in L} G(s, \pi|\mathbf{m}|^2) \right) + \left(-\frac{1}{s} + \sum'_{\mathbf{m} \in L^*} G\left(\frac{n}{2} - s, \pi|\mathbf{m}|^2\right) \right)$$

holds for $s \in \mathbb{C} \setminus \{0, \frac{n}{2}\}$, where

$$G(s, x) := \int_1^\infty t^{s-1} e^{-xt} dt, \quad \operatorname{Re} x > 0.$$

We define

$$(4.2) \quad H_n(L, s) := -\frac{1}{\frac{n}{2} - s} + \sum'_{\mathbf{m} \in L} G(s, \pi|\mathbf{m}|^2),$$

and thus the identity (4.1) becomes

$$(4.3) \quad F_n(L, s) = H_n(L, s) + H_n(L^*, \frac{n}{2} - s).$$

Hence, to be able to understand the function $F_n(L, s)$ we need first to understand the function $H_n(L, s)$. As a first step, we observe that the integral obtained by replacing the summation over L in (4.2) by integration over \mathbb{R}^n can be evaluated explicitly:

Lemma 4.1. *For each $s \in \mathbb{C}$ with $\operatorname{Re} s < \frac{n}{2}$ we have*

$$\int_{\mathbb{R}^n} G(s, \pi|\mathbf{x}|^2) d\mathbf{x} = \frac{1}{\frac{n}{2} - s}.$$

Proof. Changing to spherical coordinates we have (recalling that ω_n denotes the $(n-1)$ -dimensional volume of the unit sphere $S^{n-1} \subset \mathbb{R}^n$)

$$(4.4) \quad \begin{aligned} \int_{\mathbb{R}^n} G(s, \pi|\mathbf{x}|^2) d\mathbf{x} &= \omega_n \int_0^\infty G(s, \pi r^2) r^{n-1} dr \\ &= \frac{\omega_n}{2} \pi^{-\frac{n}{2}} \int_0^\infty G(s, x) x^{\frac{n}{2}-1} dx \\ &= \frac{\omega_n}{2} \pi^{-\frac{n}{2}} \int_0^\infty \int_1^\infty t^{s-1} e^{-xt} x^{\frac{n}{2}-1} dt dx \\ &= \frac{\omega_n}{2} \pi^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) \int_1^\infty t^{s-\frac{n}{2}-1} dt = \frac{1}{\frac{n}{2} - s}, \end{aligned}$$

where we in the last step use the well-known identity $\omega_n = 2\pi^{\frac{n}{2}} \Gamma(\frac{n}{2})^{-1}$. \square

It follows from Siegel's mean value formula [16] that the expectation value of the sum over L in (4.2) equals the integral in Lemma 4.1, and hence we have:

$$(4.5) \quad \mathbb{E}(H_n(\cdot, s)) = 0 \quad \text{for all } s \text{ with } \operatorname{Re} s < \frac{n}{2}.$$

In fact, for real s all terms in the sum in (4.2) are positive, and we will see in the proof of Theorem 1.1 that for most lattices $L \in X_n$ with n large, and $s \in (\frac{n}{4}, \frac{n}{2})$, we

have exponential cancellation between the sum and the term $-(\frac{n}{2} - s)^{-1}$: For any fixed $c \in (\frac{1}{4}, \frac{1}{2})$ there exists some $\delta > 0$ such that

$$(4.6) \quad \text{Prob}_{\mu_n} \left\{ L \in X_n \mid |H_n(L, cn)| < e^{-\delta n} \right\} \rightarrow 1$$

as $n \rightarrow \infty$. (Cf. Remark 5.2 below.) Hence the analysis of $H_n(L, s)$ is quite delicate.

The key to capturing the exponential cancellation in (4.2) and getting control on the difference $H_n(L, s)$ is our Theorem 1.3, and our starting point is to rewrite (4.2) in terms of $R_n(V)$. Note that Lemma 4.1 can be expressed as

$$\int_0^\infty G\left(s, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dV = \frac{1}{\frac{n}{2} - s}$$

(indeed, substituting $x = \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}$ in the integral we get back the second line in (4.4) above). Hence, recalling the definitions of $N_n(V)$ and $R_n(V)$ from the introduction, we have

$$(4.7) \quad H_n(L, s) = -\frac{1}{\frac{n}{2} - s} + \int_0^\infty G\left(s, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dN_n(V) = \int_0^\infty G\left(s, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dR_n(V),$$

for all s with $0 < s < \frac{n}{2}$. The idea is now that the tail of this integral will be small compared with the size of $H_n(L, s)$. The precise meaning of this statement will be clear below.

Lemma 4.2. *For $0 < x \leq s - 1$ we have*

$$x^{-s}\Gamma(s) - e^{-x} \leq G(s, x) \leq x^{-s}\Gamma(s).$$

Proof. From the definition of $G(s, x)$ we get

$$G(s, x) = \int_1^\infty t^{s-1} e^{-xt} dt = x^{-s} \int_x^\infty u^{s-1} e^{-u} du = x^{-s} \left(\Gamma(s) - \int_0^x u^{s-1} e^{-u} du \right).$$

Here, since the function $u \mapsto u^{s-1} e^{-u}$ is increasing for $u \in (0, s-1)$, we have $0 \leq \int_0^x u^{s-1} e^{-u} du \leq x^s e^{-x}$ for $0 < x \leq s-1$ and the lemma follows. \square

Applying Stirling's formula we get

$$(4.8) \quad \omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \sim \left(\frac{2\pi e}{n}\right)^{\frac{n}{2}} \left(\frac{n}{\pi}\right)^{\frac{1}{2}} \quad \text{as } n \rightarrow \infty.$$

As a consequence we note that $\pi\left(\frac{n}{\omega_n}\right)^{2/n} \sim \frac{n}{2e}$ as $n \rightarrow \infty$ and hence, for fixed $A > 0$ and all large enough n , we have $\pi\left(\frac{nA}{\omega_n}\right)^{2/n} < \frac{n}{4} - 1$. Thus, for all $c \in [\frac{1}{4}, \frac{1}{2})$ and A and n as above, Lemma 4.2 applies to give

$$(4.9) \quad \int_0^A G\left(cn, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dR_n(V) = \Gamma(cn) \pi^{-cn} \left(\frac{n}{\omega_n}\right)^{-2c} \int_0^A V^{-2c} dR_n(V) \\ + O(1) \int_0^A \exp\left\{-\pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right\} (dN_n(V) + dV),$$

with an absolute implied constant. We choose not to consider this identity for $c = \frac{1}{2}$ since in that case both the integrals in the first row of (4.9) are divergent. For notational convenience we set

$$K_{c,n} := \Gamma(cn) \pi^{-cn} \left(\frac{n}{\omega_n}\right)^{-2c}.$$

Proposition 4.3. *Let $A > 0$ be fixed. Then, for all $k < \frac{1}{2e}$, we have*

$$\text{Prob}_{\mu_n} \left\{ L \in X_n \mid \left| K_{c,n}^{-1} \int_0^A G\left(cn, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dR_n(V) - \int_0^A V^{-2c} dR_n(V) \right| < K_{c,n}^{-1} e^{-kn}, \forall c \in \left[\frac{1}{4}, \frac{1}{2}\right] \right\} \rightarrow 1$$

as $n \rightarrow \infty$.

Proof. We consider the integral with respect to $dN_n(V)$ and the integral with respect to dV separately in the error term in (4.9). Changing variables $V = \frac{\omega_n}{n} \left(\frac{x}{\pi}\right)^{n/2}$ yields

$$(4.10) \quad \int_0^A \exp\left\{-\pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right\} dV = \frac{\omega_n}{2} \pi^{-\frac{n}{2}} \int_0^{\pi(nA/\omega_n)^{2/n}} e^{-x} x^{\frac{n}{2}-1} dx.$$

Recalling that we have $\pi(nA/\omega_n)^{2/n} < k'n$ for any fixed $k' > \frac{1}{2e}$ and all sufficiently large n , as well as the fact that $x \mapsto e^{-x} x^{\frac{n}{2}-1}$ is increasing for all $0 < x < \frac{n}{2} - 1$, we find that, for $\frac{1}{2e} < k' < \frac{1}{2}$ and large enough n , (4.10) is

$$O\left(\omega_n \left(\frac{k'n}{\pi}\right)^{\frac{n}{2}} e^{-k'n}\right) = O\left(n^{\frac{1}{2}} \exp\left(\left(\frac{1}{2} \log(2ek') - k'\right)n\right)\right).$$

By taking k' sufficiently close to $\frac{1}{2e}$ it follows that for any fixed $k < \frac{1}{2e}$ there exists $n_0 \in \mathbb{Z}_{\geq 1}$ (which also depends on A) such that

$$\int_0^A \exp\left\{-\pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right\} dV < e^{-kn}$$

for all $n \geq n_0$.

Next, let $\varepsilon > 0$ be given. By possibly increasing n_0 it follows from [12, Thm. 3] (cf. also [18, Thm. 1]) that there exists $M \in \mathbb{Z}_{\geq 1}$ such that for $n \geq n_0$ we have both $N_n(A) < M$ and $N_n(M^{-1}) = 0$ with probability $> 1 - \varepsilon$. Since also $(M^{-1})^{2/n} \rightarrow 1$ as $n \rightarrow \infty$, we conclude that for any fixed constant $k < \frac{1}{2e}$ and all $n \geq n_0$ (with a possibly even larger n_0 depending on k) we have

$$\int_0^A \exp\left\{-\pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right\} dN_n(V) < M \exp\left\{-\pi\left(\frac{nM^{-1}}{\omega_n}\right)^{\frac{2}{n}}\right\} < e^{-kn},$$

with probability $> 1 - \varepsilon$. Hence for our fixed $A > 0$ and $k < \frac{1}{2e}$ and all $n \geq n_0$, the absolute error in (4.9) is $< Ce^{-kn}$, where C is an absolute constant, with probability $> 1 - \varepsilon$. Thus for any $k' < k$ the absolute error is $< e^{-k'n}$ for all sufficiently large n with probability $> 1 - \varepsilon$, and the proposition follows. \square

Remark 4.4. We stress that with an appropriate choice of k , the upper bound $K_{c,n}^{-1} e^{-kn}$ in Proposition 4.3 tends to zero as $n \rightarrow \infty$, uniformly with respect to $c \in [\frac{1}{4}, \frac{1}{2}]$. Indeed, note that

$$\begin{aligned} K_{c,n}^{-1} e^{-kn} &= O\left(n^{\frac{1}{2}} \left(\frac{\pi e}{cn}\right)^{cn} n^{2c} \left(\frac{n}{2\pi e}\right)^{cn} n^{-c} e^{-kn}\right) \\ &= O\left(n^{\frac{1}{2}+c} \exp\left(-(k + c \log(2c))n\right)\right). \end{aligned}$$

Here $k + c \log(2c) \geq k - \frac{1}{4} \log 2 = k - 0.1732\dots$ for all $c \in [\frac{1}{4}, \frac{1}{2}]$, and the last difference is positive when k is sufficiently close to $\frac{1}{2e} = 0.1839\dots$

Next we estimate the tail of the integral giving $H_n(L, s)$, normalized in the same way as the integral in Proposition 4.3. The proof is similar to the proof of Lemma 2.6. We first recall two bounds on $G(s, x)$ which will be used several times in this paper.

Lemma 4.5. *The following bound holds uniformly for all $x > 0$, $s \geq 1$,*

$$G(s, x) \ll s^{-\frac{1}{2}} \left(\frac{ex}{s} \right)^{-s}.$$

In the case $x \geq s \geq 1$ we also have the stronger bound

$$G(s, x) \ll s^{-\frac{1}{2}} e^{-x}.$$

Proof. Cf. [15, Cor. 2]. □

Lemma 4.6. *Let $c_1 \in (\frac{1}{4}, \frac{1}{2})$. Then, for all $\varepsilon > 0$ there exist constants $A_0 > 0$ and $n_0 \in \mathbb{Z}_{\geq 3}$ such that for all $A \geq A_0$ and $n \geq n_0$ we have*

$$\text{Prob}_{\mu_n} \left\{ L \in X_n \mid \sup_{c \in [c_1, \frac{1}{2}]} \left| K_{c,n}^{-1} \int_A^\infty G\left(cn, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dR_n(V) \right| \leq \varepsilon \right\} \geq 1 - \varepsilon.$$

Proof. Let $\varepsilon > 0$ and $\delta \in (0, 2c_1 - \frac{1}{2})$ be given. It follows from Theorem 1.3 that for each $n \geq 3$ there exists a set $X'_n \subset X_n$ with $\mu_n(X'_n) \geq 1 - \varepsilon$ such that for all $L \in X'_n$ and all $V \geq 10$ we have $|R_n(V)| \ll_{\varepsilon, \delta} V^{\frac{1}{2} + \delta}$, where the implied constant is independent of n and L . Now, integrating by parts and using $\frac{\partial}{\partial x} G(s, x) = -G(s+1, x)$, we have

$$(4.11) \quad \int_A^\infty G\left(cn, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dR_n(V) = \left[G\left(cn, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) R_n(V) \right]_{V=A}^{V=\infty} + \frac{2\pi}{n} \left(\frac{n}{\omega_n}\right)^{\frac{2}{n}} \int_A^\infty G\left(cn+1, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) V^{\frac{2}{n}-1} R_n(V) dV.$$

Hence, using Lemma 4.5 we get, for any $L \in X'_n$ (with n sufficiently large) and all $A \geq 10$,

$$\begin{aligned} & \left| K_{c,n}^{-1} \int_A^\infty G\left(cn, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dR_n(V) \right| \\ & \ll_{\varepsilon, \delta} A^{-2c+\frac{1}{2}+\delta} + \left(\frac{cn+1}{cn}\right)^{cn} \int_A^\infty V^{-2c-\frac{1}{2}+\delta} dV \ll_{\varepsilon, \delta, c_1} A^{-2c_1+\frac{1}{2}+\delta}, \end{aligned}$$

uniformly over all $c \in [c_1, \frac{1}{2}]$. Thus we can make the left hand side above as small as we like, by choosing A large enough. □

Given $\varepsilon > 0$ and $c_1 \in (\frac{1}{4}, \frac{1}{2})$, it follows from (4.7), Proposition 4.3 and Lemma 4.6 that there exists $A_0 > 0$ such that for all $A \geq A_0$ there exists $n_0 \in \mathbb{Z}_{\geq 3}$ such that for all $n \geq n_0$ we have

$$(4.12) \quad \text{Prob}_{\mu_n} \left\{ L \in X_n \mid \sup_{c \in [c_1, \frac{1}{2}]} \left| K_{c,n}^{-1} H_n(L, cn) - \int_0^A V^{-2c} dR_n(V) \right| \leq \varepsilon \right\} \geq 1 - \varepsilon.$$

Since our goal is to understand the function $F_n(L, cn)$ for $c \in [c_1, \frac{1}{2})$ it remains to study $J_n(L, s) := H_n(L, \frac{n}{2} - s)$ for $s = cn$ with $c \in [c_1, \frac{1}{2})$ (recall (4.3)).

Proposition 4.7. *Given any $c_1 \in (\frac{1}{4}, \frac{1}{2})$ there exists a constant $k > 0$ such that*

$$\text{Prob}_{\mu_n} \left\{ L \in X_n \mid |K_{c,n}^{-1} J_n(L, cn)| < e^{-kn}, \forall c \in [c_1, \frac{1}{2}] \right\} \rightarrow 1$$

as $n \rightarrow \infty$.

Proof. It follows from (4.7) and integration by parts, together with the estimates in Lemma 4.5 and the bound $G(s, x) \ll x^{-1}e^{-x}$ for $0 \leq s \leq 1$, that

$$\begin{aligned} J_n(L, cn) &= \int_0^\infty G\left(\frac{n}{2} - cn, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dR_n(V) \\ &= \frac{2\pi}{n} \left(\frac{n}{\omega_n}\right)^{\frac{2}{n}} \int_0^\infty G\left(\frac{n}{2} - cn + 1, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) V^{\frac{2}{n}-1} R_n(V) dV \end{aligned}$$

(cf. (4.11)). Furthermore, changing variables $V = \frac{\omega_n}{n} \left(\frac{x}{\pi}\right)^{n/2}$ we obtain

$$(4.13) \quad J_n(L, cn) = \int_0^\infty G\left(\frac{n}{2} - cn + 1, x\right) R_n\left(\frac{\omega_n}{n} \left(\frac{x}{\pi}\right)^{\frac{n}{2}}\right) dx.$$

Given $\varepsilon, \delta \in (0, \frac{1}{2})$, it follows from [18] and Theorem 1.3 that there exist $n_0 \in \mathbb{Z}_{\geq 3}$, $M \in \mathbb{Z}_{\geq 1}$ and sets $X_n'' \subset X_n$ with $\mu_n(X_n'') > 1 - \varepsilon$ such that for all $n \geq n_0$ and $L \in X_n''$ we have $N_n(V) = 0$ for all $V \in [0, M^{-1}]$, $N_n(10) < M$, and $|R_n(V)| \ll_{\varepsilon, \delta} V^{\frac{1}{2} + \delta}$ for all $V \geq 10$. It follows that, for all $n \geq n_0$ and $L \in X_n''$, we have

$$|R_n(V)| \ll_{\varepsilon, \delta} \min(V, V^{\frac{1}{2} + \delta}), \quad \forall V \geq 0.$$

We now estimate $J_n(L, cn)$ for all $c \in [\frac{1}{4}, \frac{1}{2}]$ and $L \in X_n''$ ($n \geq n_0$) by splitting the integral in (4.13) into two parts. More precisely, for $n \geq n_0$ and $L \in X_n''$, we have

$$(4.14) \quad |J_n(L, cn)| \ll \int_0^{W_n} G\left(\frac{n}{2} - cn + 1, x\right) \frac{\omega_n}{n} \left(\frac{x}{\pi}\right)^{\frac{n}{2}} dx \\ + \int_{W_n}^\infty G\left(\frac{n}{2} - cn + 1, x\right) \left(\frac{\omega_n}{n} \left(\frac{x}{\pi}\right)^{\frac{n}{2}}\right)^{\frac{1}{2} + \delta} dx,$$

where $W_n = \pi\left(\frac{n}{\omega_n}\right)^{\frac{2}{n}}$. In (4.14) and in all other " \ll " bounds below, the implied constant may depend on ε, δ , but is independent of n, L, c (subject to $c \in [\frac{1}{4}, \frac{1}{2}]$). Recall here that $W_n \sim \frac{n}{2e}$ as $n \rightarrow \infty$. We call the integrals in (4.14) I_1 and I_2 respectively.

To begin with we set $T_n = \min(\frac{n}{2} - cn + 1, W_n)$ and use Lemma 4.5 to get

$$(4.15) \quad I_1 \ll \frac{\omega_n}{n} \pi^{-\frac{n}{2}} \left(\frac{n}{2} - cn + 1\right)^{(\frac{1}{2}-c)n + \frac{1}{2}} e^{(c-\frac{1}{2})n} \int_0^{T_n} x^{cn-1} dx \\ + \frac{\omega_n}{n} \pi^{-\frac{n}{2}} \left(\frac{n}{2} - cn + 1\right)^{-\frac{1}{2}} \int_{T_n}^{W_n} x^{\frac{n}{2}} e^{-x} dx.$$

When $T_n = W_n$ the second integral in (4.15) vanishes and we have

$$(4.16) \quad I_1 \ll \frac{\omega_n}{n^2} \pi^{-\frac{n}{2}} \left(\frac{n}{2} - cn + 1\right)^{(\frac{1}{2}-c)n + \frac{1}{2}} e^{(c-\frac{1}{2})n} W_n^{cn} \\ \ll n^{-1} \left(\frac{\omega_n}{n}\right)^{1-2c} (\pi e)^{(c-\frac{1}{2})n} \left(\frac{n}{2} - cn + 1\right)^{(\frac{1}{2}-c)n + \frac{1}{2}}.$$

On the other hand, when $T_n = \frac{n}{2} - cn + 1$ we have

(4.17)

$$I_1 \ll \frac{\omega_n}{n^2} \pi^{-\frac{n}{2}} e^{(c-\frac{1}{2})n} \left(\frac{n}{2} - cn + 1\right)^{\frac{n}{2} + \frac{1}{2}} + \frac{\omega_n}{n} \pi^{-\frac{n}{2}} \left(\frac{n}{2} - cn + 1\right)^{-\frac{1}{2}} \int_0^{W_n} x^{\frac{n}{2}} e^{-x} dx.$$

One checks that $x \mapsto x^{\frac{n}{2}} e^{-x}$ is increasing for all $0 < x < \frac{n}{2}$. In addition $W_n < \frac{n}{2}$ for all large enough n . Hence after possibly increasing n_0 , we have that for all $n \geq n_0$, (4.17) is

$$(4.18) \quad \begin{aligned} &\ll \frac{\omega_n}{n^2} \pi^{-\frac{n}{2}} e^{(c-\frac{1}{2})n} \left(\frac{n}{2} - cn + 1\right)^{\frac{n}{2} + \frac{1}{2}} + \frac{\omega_n}{n} \pi^{-\frac{n}{2}} \left(\frac{n}{2} - cn + 1\right)^{-\frac{1}{2}} W_n^{\frac{n}{2} + 1} e^{-W_n} \\ &\ll \frac{\omega_n}{n^2} \pi^{-\frac{n}{2}} e^{(c-\frac{1}{2})n} \left(\frac{n}{2} - cn + 1\right)^{\frac{n}{2} + \frac{1}{2}} + \omega_n^{-\frac{2}{n}} \left(\frac{n}{2} - cn + 1\right)^{-\frac{1}{2}} e^{-W_n}. \end{aligned}$$

Next we estimate the integral I_2 . We set $S_n = \max(\frac{n}{2} - cn + 1, W_n)$ and use Lemma 4.5 to get

$$(4.19) \quad \begin{aligned} I_2 &\ll \left(\frac{\omega_n}{n}\right)^{\frac{1}{2} + \delta} \pi^{-(\frac{1}{4} + \frac{\delta}{2})n} \left(\frac{n}{2} - cn + 1\right)^{(\frac{1}{2} - c)n + \frac{1}{2}} e^{(c-\frac{1}{2})n} \int_{W_n}^{S_n} x^{(c-\frac{1}{4} + \frac{\delta}{2})n-1} dx \\ &\quad + \left(\frac{\omega_n}{n}\right)^{\frac{1}{2} + \delta} \pi^{-(\frac{1}{4} + \frac{\delta}{2})n} \left(\frac{n}{2} - cn + 1\right)^{-\frac{1}{2}} \int_{S_n}^{\infty} x^{(\frac{1}{4} + \frac{\delta}{2})n} e^{-x} dx. \end{aligned}$$

When $S_n = W_n$ the first integral in (4.19) vanishes and we obtain, estimating the function $g(x) = x^{(\frac{1}{4} + \frac{\delta}{2})n+3} e^{-x}$ with its maximum,

(4.20)

$$\begin{aligned} I_2 &\ll \left(\frac{\omega_n}{n}\right)^{\frac{1}{2} + \delta} \pi^{-(\frac{1}{4} + \frac{\delta}{2})n} \left(\frac{n}{2} - cn + 1\right)^{-\frac{1}{2}} \int_{W_n}^{\infty} x^{(\frac{1}{4} + \frac{\delta}{2})n+3} e^{-x} \frac{dx}{x^3} \\ &\ll \left(\frac{\omega_n}{n}\right)^{\frac{1}{2} + \delta} W_n^{-2} \pi^{-(\frac{1}{4} + \frac{\delta}{2})n} \left(\frac{n}{2} - cn + 1\right)^{-\frac{1}{2}} \left(\left(\frac{1}{4} + \frac{\delta}{2}\right)n + 3\right)^{(\frac{1}{4} + \frac{\delta}{2})n+3} e^{-(\frac{1}{4} + \frac{\delta}{2})n} \\ &\ll \left(\frac{\omega_n}{n}\right)^{\frac{1}{2} + \delta + \frac{4}{n}} \pi^{-(\frac{1}{4} + \frac{\delta}{2})n} \left(\frac{n}{2} - cn + 1\right)^{-\frac{1}{2}} \left(\left(\frac{1}{4} + \frac{\delta}{2}\right)n + 3\right)^{(\frac{1}{4} + \frac{\delta}{2})n+3} e^{-(\frac{1}{4} + \frac{\delta}{2})n}. \end{aligned}$$

In the remaining case, that is $S_n = \frac{n}{2} - cn + 1$, we have

(4.21)

$$\begin{aligned} I_2 &\ll \left(\frac{\omega_n}{n}\right)^{\frac{1}{2} + \delta} \pi^{-(\frac{1}{4} + \frac{\delta}{2})n} \left(\frac{n}{2} - cn + 1\right)^{(\frac{1}{2} - c)n + \frac{1}{2}} e^{(c-\frac{1}{2})n} \int_0^{\frac{n}{2} - cn + 1} x^{(c-\frac{1}{4} + \frac{\delta}{2})n-1} dx \\ &\quad + \left(\frac{\omega_n}{n}\right)^{\frac{1}{2} + \delta} \pi^{-(\frac{1}{4} + \frac{\delta}{2})n} \left(\frac{n}{2} - cn + 1\right)^{-\frac{5}{2}} \left(\left(\frac{1}{4} + \frac{\delta}{2}\right)n + 3\right)^{(\frac{1}{4} + \frac{\delta}{2})n+3} e^{-(\frac{1}{4} + \frac{\delta}{2})n} \\ &\ll n^{-1} \left(\frac{\omega_n}{n}\right)^{\frac{1}{2} + \delta} \pi^{-(\frac{1}{4} + \frac{\delta}{2})n} e^{(c-\frac{1}{2})n} \left(\frac{n}{2} - cn + 1\right)^{(\frac{1}{4} + \frac{\delta}{2})n + \frac{1}{2}} \\ &\quad + \left(\frac{\omega_n}{n}\right)^{\frac{1}{2} + \delta} \pi^{-(\frac{1}{4} + \frac{\delta}{2})n} \left(\frac{n}{2} - cn + 1\right)^{-\frac{5}{2}} \left(\left(\frac{1}{4} + \frac{\delta}{2}\right)n + 3\right)^{(\frac{1}{4} + \frac{\delta}{2})n+3} e^{-(\frac{1}{4} + \frac{\delta}{2})n}. \end{aligned}$$

(Recall that the implied constant is allowed to depend on δ .)

Collecting the results in (4.16), (4.18), (4.20) and (4.21) we get, for all $n \geq n_0$, $L \in X_n''$ and $c \in [\frac{1}{4}, \frac{1}{2}]$,

(4.22)

$$\begin{aligned} |J_n(L, cn)| &\ll n^{-1} \left(\frac{\omega_n}{n} \right)^{1-2c} (\pi e)^{(c-\frac{1}{2})n} \left(\frac{n}{2} - cn + 1 \right)^{(\frac{1}{2}-c)n+\frac{1}{2}} \\ &\quad + n^{-1} \left(\frac{\omega_n}{n} \right)^{\frac{1}{2}+\delta} \pi^{-(\frac{1}{4}+\frac{\delta}{2})n} e^{(c-\frac{1}{2})n} \left(\frac{n}{2} - cn + 1 \right)^{(\frac{1}{4}+\frac{\delta}{2})n+\frac{1}{2}} \\ &\quad + \left(\frac{\omega_n}{n} \right)^{\frac{1}{2}+\delta} \pi^{-(\frac{1}{4}+\frac{\delta}{2})n} \left(\frac{n}{2} - cn + 1 \right)^{-\frac{5}{2}} \left(\left(\frac{1}{4} + \frac{\delta}{2} \right) n + 3 \right)^{(\frac{1}{4}+\frac{\delta}{2})n+3} e^{-(\frac{1}{4}+\frac{\delta}{2})n} \end{aligned}$$

when $W_n \leq \frac{n}{2} - cn + 1$, and

(4.23)

$$\begin{aligned} |J_n(L, cn)| &\ll \frac{\omega_n}{n^2} \pi^{-\frac{n}{2}} e^{(c-\frac{1}{2})n} \left(\frac{n}{2} - cn + 1 \right)^{\frac{n}{2}+\frac{1}{2}} + \omega_n^{-\frac{2}{n}} \left(\frac{n}{2} - cn + 1 \right)^{-\frac{1}{2}} e^{-W_n} \\ &\quad + \left(\frac{\omega_n}{n} \right)^{\frac{1}{2}+\delta+\frac{4}{n}} \pi^{-(\frac{1}{4}+\frac{\delta}{2})n} \left(\frac{n}{2} - cn + 1 \right)^{-\frac{1}{2}} \left(\left(\frac{1}{4} + \frac{\delta}{2} \right) n + 3 \right)^{(\frac{1}{4}+\frac{\delta}{2})n+3} e^{-(\frac{1}{4}+\frac{\delta}{2})n} \end{aligned}$$

when $\frac{n}{2} - cn + 1 \leq W_n$.

It now remains to prove that all terms in (4.22) and (4.23) are as small as the proposition claims. We will prove that there exists a constant $k > 0$ such that if δ has been fixed to be sufficiently small (as depends only on c_1), then for all sufficiently large n we have $K_{c,n}^{-1} |J_n(L, cn)| \ll n^4 e^{-2kn}$ for all $c \in [c_1, \frac{1}{2}]$ and $L \in X_n''$. Hence, a fortiori, $K_{c,n}^{-1} |J_n(L, cn)| < e^{-kn}$ for n large enough, and this completes the proof.

We first consider (4.22). Using Stirling's formula and (4.8) we get

$$\begin{aligned} K_{c,n}^{-1} n^{-1} \left(\frac{\omega_n}{n} \right)^{1-2c} (\pi e)^{(c-\frac{1}{2})n} \left(\frac{n}{2} - cn + 1 \right)^{(\frac{1}{2}-c)n+\frac{1}{2}} \\ \ll n^{2c-\frac{1}{2}} \left(\frac{1}{2} - c + \frac{1}{n} \right)^{\frac{1}{2}} \exp(-f_n(c) \cdot n), \end{aligned}$$

where

$$f_n(c) = c \log c + (2c - \frac{1}{2}) \log 2 + (c - \frac{1}{2}) \log \left(\frac{1}{2} - c + \frac{1}{n} \right).$$

Using $W_n \sim \frac{n}{2e}$ and $\frac{1}{2} - \frac{1}{2e} = 0.316\dots$ we find that for n sufficiently large the assumption $W_n \leq \frac{n}{2} - cn + 1$ implies $c \leq 0.32$. Moreover, for all $c \in [\frac{1}{4}, 0.32]$ we have

$$\begin{aligned} (4.24) \quad f'_n(c) &= \log c + 2 + 2 \log 2 + \log \left(\frac{1}{2} - c + \frac{1}{n} \right) - \left(\frac{n}{2} - cn + 1 \right)^{-1} \\ &\geq 2 + \log \left(\frac{1}{2} - 0.32 \right) - \left(\frac{n}{2} - 0.32n \right)^{-1}, \end{aligned}$$

which is positive for n sufficiently large. Hence for n sufficiently large and for all $c \in [c_1, \frac{1}{2}]$ satisfying $W_n \leq \frac{n}{2} - cn + 1$, we have (writing $f_\infty(c) := \lim_{n \rightarrow \infty} f_n(c)$ and noticing that the computation in (4.24) also proves $f'_\infty > 0$ for $c \in [\frac{1}{4}, 0.32]$):

$$f_n(c) \geq f_n(c_1) > \frac{1}{2} f_\infty(c_1) > 0.$$

Hence the first term in (4.22) is small enough. Continuing, we find that

$$\begin{aligned} K_{c,n}^{-1} n^{-1} \left(\frac{\omega_n}{n} \right)^{\frac{1}{2}+\delta} \pi^{-(\frac{1}{4}+\frac{\delta}{2})n} e^{(c-\frac{1}{2})n} \left(\frac{n}{2} - cn + 1 \right)^{(\frac{1}{4}+\frac{\delta}{2})n+\frac{1}{2}} \\ \ll n^{c-\frac{\delta}{2}-\frac{1}{4}} \left(\frac{1}{2} - c + \frac{1}{n} \right)^{\frac{1}{2}} \exp(-g_n(c) \cdot n), \end{aligned}$$

where

$$g_n(c) = c \log c + \left(c - \frac{1}{4} - \frac{\delta}{2}\right) \log 2 + \left(\frac{1}{4} - \frac{\delta}{2} - c\right) - \left(\frac{1}{4} + \frac{\delta}{2}\right) \log \left(\frac{1}{2} - c + \frac{1}{n}\right).$$

Here $g'_n(c) > \log(2c) + (2 - 4c + \frac{4}{n})^{-1} > 0$ for all $n \geq 10$ and $c \in [\frac{1}{4}, \frac{1}{2}]$; hence $g_n(c) \geq g_n(c_1)$ for all $c \in [c_1, \frac{1}{2}]$. Thus, since for all sufficiently large n and small δ we have that $g_n(c_1)$ is larger than a positive constant which only depends on c_1 , the second term in (4.22) is small enough. Next we note that

$$(4.25) \quad K_{c,n}^{-1} \left(\frac{\omega_n}{n}\right)^{\frac{1}{2}+\delta} \pi^{-(\frac{1}{4}+\frac{\delta}{2})n} \left(\frac{n}{2} - cn + 1\right)^{-\frac{5}{2}} \left(\left(\frac{1}{4} + \frac{\delta}{2}\right)n + 3\right)^{(\frac{1}{4}+\frac{\delta}{2})n+3} e^{-(\frac{1}{4}+\frac{\delta}{2})n} \\ \ll n^{c-\frac{\delta}{2}+\frac{3}{4}} \left(\frac{1}{2} - c + \frac{1}{n}\right)^{-\frac{5}{2}} \exp(-h_n(c) \cdot n),$$

where

$$h_n(c) = c \log c + \left(c - \frac{1}{4} - \frac{\delta}{2}\right) \log 2 - \left(\frac{1}{4} + \frac{\delta}{2}\right) \log \left(\frac{1}{4} + \frac{\delta}{2} + \frac{3}{n}\right).$$

Now $h'_n(c) \geq 1 - \log 2 > 0$ for all $c \geq \frac{1}{4}$, independently of n and δ , and thus $h_n(c) \geq h_n(c_1)$ for all $c \in [c_1, \frac{1}{2}]$; also for all sufficiently large n and small δ we have that $h_n(c_1)$ is larger than a positive constant which only depends on c_1 . Thus the third term in (4.22) is small enough.

We now give a similar treatment of the terms in (4.23). First we observe that

$$K_{c,n}^{-1} \frac{\omega_n}{n^2} \pi^{-\frac{n}{2}} e^{(c-\frac{1}{2})n} \left(\frac{n}{2} - cn + 1\right)^{\frac{n}{2}+\frac{1}{2}} \ll n^{c-\frac{1}{2}} \left(\frac{1}{2} - c + \frac{1}{n}\right)^{\frac{1}{2}} \exp(-j_n(c) \cdot n),$$

where

$$j_n(c) = c \log c + \left(c - \frac{1}{2}\right) \log 2 - c - \frac{1}{2} \log \left(\frac{1}{2} - c + \frac{1}{n}\right).$$

Note that $j'_n(c) = \log(2c) + (1 - 2c + \frac{2}{n})^{-1} > 0$ for all $c \in [\frac{1}{4}, \frac{1}{2}]$ and all $n \geq 3$. Furthermore, using $\frac{1}{2} - \frac{1}{2e} = 0.316\dots$, it follows that for n sufficiently large the assumption $W_n \geq \frac{n}{2} - cn + 1$ implies $c \geq 0.3$. Hence $j_n(c) \geq j_n(0.3)$, and for all $n \geq 1000$ we have $j_n(0.3) \geq j_{1000}(0.3) = 0.00240\dots > 0$. Hence the first term in (4.23) is as small as desired. Next we note that, for all sufficiently large n such that $W_n > (\frac{1}{2e} - \delta)n$, we have

$$K_{c,n}^{-1} \omega_n^{-\frac{2}{n}} \left(\frac{n}{2} - cn + 1\right)^{-\frac{1}{2}} e^{-W_n} \ll K_{c,n}^{-1} n^{\frac{1}{2}} \left(\frac{1}{2} - c + \frac{1}{n}\right)^{-\frac{1}{2}} e^{-(\frac{1}{2e}-\delta)n}.$$

Hence it follows from Remark 4.4 that also the second term in (4.23) is as small as desired. Finally, since the third term in (4.23) differs from the the third term in (4.22) only by a factor of polynomial size in n , the treatments of these terms are almost identical. Note in particular that the exponential decay in (4.25) is uniform for $c \in [c_1, \frac{1}{2}]$. This concludes the proof of the proposition. \square

Remark 4.8. Recall from (4.3) that we are interested in $J_n(L^*, cn)$. Since the measure μ_n is invariant under the homeomorphism $L \mapsto L^*$ of X_n onto itself, we have the following consequence of Proposition 4.7: Given any $c_1 \in (\frac{1}{4}, \frac{1}{2})$ there exists a constant $k > 0$ such that

$$\text{Prob}_{\mu_n} \left\{ L \in X_n \mid K_{c,n}^{-1} |J_n(L^*, cn)| < e^{-kn}, \forall c \in [c_1, \frac{1}{2}] \right\} \rightarrow 1$$

as $n \rightarrow \infty$.

We collect the results of this section in the following theorem.

Theorem 4.9. *Let $c_1 \in (\frac{1}{4}, \frac{1}{2})$. Then for all $\varepsilon > 0$ there exists $A_0 > 0$ such that for all $A \geq A_0$ there exists $n_0 \in \mathbb{Z}_{\geq 3}$ such that for all $n \geq n_0$ we have*

$$\text{Prob}_{\mu_n} \left\{ L \in X_n \mid \sup_{c \in [c_1, \frac{1}{2})} \left| V_n^{-2c} E_n(L, cn) - \int_0^A V^{-2c} dR_n(V) \right| \leq \varepsilon \right\} \geq 1 - \varepsilon.$$

Proof. Recall that $V_n = \frac{\omega_n}{n}$. Since

$$K_{c,n}^{-1} F_n(L, cn) = V_n^{-2c} E_n(L, cn)$$

the theorem follows from Proposition 4.3, Lemma 4.6 (cf. (4.12)) and Remark 4.8. \square

5. PROOF OF THEOREM 1.1

Theorem 4.9 says that for $c \in [c_1, \frac{1}{2})$ the random variable $\int_0^A V^{-2c} dR_n(V)$ is, with large probability, uniformly close to the (normalized) Epstein zeta function provided that A and n are appropriately large. We now show that this random variable is close in distribution to the corresponding truncation of $H(c)$.

Lemma 5.1. *Let $\frac{1}{4} < c_1 < c_2 < \frac{1}{2}$ and $A > 0$ be fixed. Then the $C([c_1, c_2])$ -valued random function*

$$c \mapsto \int_0^A V^{-2c} dR_n(V)$$

converges in distribution to the random function

$$c \mapsto \int_0^A V^{-2c} dR(V)$$

as $n \rightarrow \infty$.

Proof. Expressed in more explicit terms, recalling the definitions of $R_n(V)$ and $R(V)$ (see (1.6) and (1.9)), we need to prove that the random function

$$c \mapsto 2 \sum_{\mathcal{V}_j \leq A} \mathcal{V}_j^{-2c} - \frac{A^{1-2c}}{1-2c}$$

converges in distribution to

$$c \mapsto 2 \sum_{T_j \leq A} T_j^{-2c} - \frac{A^{1-2c}}{1-2c}$$

as $n \rightarrow \infty$. Note that the function f_A defined in (2.13), considered as a function from $\Omega \setminus \Omega^{(\infty)}$ into $C([c_1, c_2])$, is continuous on the open set $\cup_{j=0}^{\infty} (\Omega^{(j)})^\circ$ (cf. (2.14), (2.15)), which has full (**P**-)measure in Ω . Now the lemma follows from [18, Thm. 1'] and [5, Thm. 2.7]. \square

We let $\mathcal{P}(C([c_1, c_2]))$ denote the set of Borel probability measures on $C([c_1, c_2])$. We recall that for $P, Q \in \mathcal{P}(C([c_1, c_2]))$ the Lévy-Prohorov distance $\pi(P, Q)$ between P and Q is defined as

(5.1)

$$\pi(P, Q) := \inf \left\{ \varepsilon > 0 \mid P(B) \leq Q(B^\varepsilon) + \varepsilon \text{ for all Borel sets } B \subseteq C([c_1, c_2]) \right\},$$

where B^ε is the open ε -neighbourhood of B in $C([c_1, c_2])$ (cf. [5]). Since $C([c_1, c_2])$ is separable, it is known that convergence in the metric π is equivalent to weak convergence in $\mathcal{P}(C([c_1, c_2]))$.

Proof of Theorem 1.1. Let $\varepsilon > 0$ be given and let μ_{E_n} , $\mu_{E_{n,A}}$, μ_{H_A} and μ_H be the distributions of the $C([c_1, c_2])$ -valued random functions $c \mapsto V_n^{-2c} E_n(\cdot, cn)$, $c \mapsto \int_0^A V^{-2c} dR_n(V)$, $c \mapsto \int_0^A V^{-2c} dR(V)$ and $c \mapsto H(c)$, respectively. Let further $A > 0$ and $n_0 \in \mathbb{Z}_{\geq 3}$ be large enough for Theorem 4.9, Lemma 5.1 and Lemma 2.6 to guarantee that $\pi(\mu_{E_n}, \mu_{E_{n,A}}) \leq \varepsilon$, $\pi(\mu_{E_{n,A}}, \mu_{H_A}) \leq \varepsilon$ and $\pi(\mu_{H_A}, \mu_H) \leq \varepsilon$ hold for all $n \geq n_0$. It follows from the triangle inequality that $\pi(\mu_{E_n}, \mu_H) \leq 3\varepsilon$ for all $n \geq n_0$. We conclude that μ_{E_n} converges (in the metric π) to μ_H as $n \rightarrow \infty$ and the theorem follows. \square

Remark 5.2. We note that our claim in (4.6) about exponential cancellation in $H_n(L, cn)$ follows easily from (4.12) and Lemma 5.1. Indeed, given $\varepsilon > 0$ we choose $A > 0$ and $n_0 \in \mathbb{Z}_{\geq 3}$ such that (4.12) holds for all $n \geq n_0$, and using Lemma 5.1 we see that there exists some $M > 0$ and $n'_0 \in \mathbb{Z}_{>0}$ such that for all $n \geq n'_0$ we have $|\int_0^A V^{-2c} dR_n(V)| < M$ for our fixed $c \in (\frac{1}{4}, \frac{1}{2})$, with (μ_n) -probability $\geq 1 - \varepsilon$. It follows that

$$\text{Prob}_{\mu_n} \left\{ L \in X_n \mid |H_n(L, cn)| < (M + \varepsilon) K_{c,n} \right\} \geq 1 - 2\varepsilon$$

for all $n \geq \max(n_0, n'_0)$. But $K_{c,n} \ll (2c)^{cn} n^{-(c+\frac{1}{2})}$ as $n \rightarrow \infty$, and thus for all sufficiently large n we have $(M + \varepsilon) K_{c,n} < e^{-\delta n}$, where $\delta := -c \log(2c) > 0$. Since $\varepsilon > 0$ was arbitrary, this concludes the proof of (4.6).

6. AN EXTENSION OF THEOREM 1.1 AND PROOFS OF THEOREM 1.4 AND COROLLARY 1.5

In this section we are interested in extending the result in Theorem 1.1 to the case $c_2 = \frac{1}{2}$. The problem is that neither $E_n(L, cn)$ nor $H(c)$ is defined for $c = \frac{1}{2}$. We overcome this problem by subtracting the singular part of $E_n(L, cn)$ from $E_n(L, cn)$ and $H(c)$. For the rest of this section we let $c_1 \in (\frac{1}{4}, \frac{1}{2})$ be fixed.

Recall that $E_n(L, s)$ has a simple pole at $s = \frac{n}{2}$ with residue $\pi^{\frac{n}{2}} \Gamma(\frac{n}{2})^{-1}$. Hence, for all n and all $L \in X_n$, the limit

$$\lim_{c \rightarrow \frac{1}{2}} \left(E_n(L, cn) - \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})(cn - \frac{n}{2})} \right)$$

exists. Now, since

$$\lim_{c \rightarrow \frac{1}{2}} V_n^{-2c} = \frac{n}{\omega_n} = \frac{n \Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}},$$

basic complex analysis gives that also the limit

$$(6.1) \quad \lim_{c \rightarrow \frac{1}{2}} \left(V_n^{-2c} E_n(L, cn) + \frac{1}{1 - 2c} \right)$$

exists for all n and all $L \in X_n$. Hence we can consider

$$c \mapsto \widehat{E}_n(\cdot, cn) := V_n^{-2c} E_n(\cdot, cn) + \frac{1}{1 - 2c}$$

as a $C([c_1, \frac{1}{2}])$ -valued random function. Here, of course, the value of the function at $c = \frac{1}{2}$ is given by the limit (6.1). We now have the following immediate corollary of Theorem 4.9.

Corollary 6.1. *Let $c_1 \in (\frac{1}{4}, \frac{1}{2})$. Then for all $\varepsilon > 0$ there exists $A_0 > 1$ such that for all $A \geq A_0$ there exists $n_0 \in \mathbb{Z}_{\geq 3}$ such that for all $n \geq n_0$ we have*

$$\text{Prob}_{\mu_n} \left\{ L \in X_n \mid \sup_{c \in [c_1, \frac{1}{2}]} \left| \widehat{E}_n(L, cn) - \left(\int_0^1 V^{-2c} dN_n(V) + \int_1^A V^{-2c} dR_n(V) \right) \right| \leq \varepsilon \right\} \geq 1 - \varepsilon.$$

Proof. Note that

$$\int_0^A V^{-2c} dR_n(V) + \frac{1}{1-2c} = \int_0^1 V^{-2c} dN_n(V) + \int_1^A V^{-2c} dR_n(V)$$

for all $c \in [c_1, \frac{1}{2})$. Hence the corollary follows from Theorem 4.9 since both $\widehat{E}_n(L, cn)$ and $\int_0^1 V^{-2c} dN_n(V) + \int_1^A V^{-2c} dR_n(V)$ are continuous on $[c_1, \frac{1}{2}]$, for each fixed $L \in X_n$. \square

We set

$$\widehat{H}(c) := \int_0^\infty V^{-2c} dR(V) + \frac{1}{1-2c} \text{ for } c \in [c_1, \frac{1}{2}) \text{ and } \widehat{H}(\frac{1}{2}) := Z_0.$$

It follows from Lemma 2.4, Lemma 2.7, Lemma 2.9, Remark 2.10 and [5, p. 84] that we can consider $c \mapsto \widehat{H}(c)$ as a $C([c_1, \frac{1}{2}])$ -valued random function on Ω (cf. Remark 2.5). Furthermore we note that (2.17) and Remark 2.11 give, for all $c \in [c_1, \frac{1}{2}]$, the formula

$$\widehat{H}(c) = \int_0^1 V^{-2c} dN(V) + \int_1^\infty V^{-2c} dR(V).$$

We are now ready to prove the following extension of Theorem 1.1.

Theorem 6.2. *Let $c_1 \in (\frac{1}{4}, \frac{1}{2})$. Then the distribution of the $C([c_1, \frac{1}{2}])$ -valued random function $c \mapsto \widehat{E}_n(\cdot, cn)$ converges to the distribution of $c \mapsto \widehat{H}(c)$ as $n \rightarrow \infty$.*

Proof. Let $h_A \in C([c_1, \frac{1}{2}])$ be given by

$$h_A(c) = \begin{cases} \frac{1-A^{1-2c}}{1-2c} & \text{if } c \in [c_1, \frac{1}{2}), \\ -\log A & \text{if } c = \frac{1}{2}. \end{cases}$$

To begin with we note that the function $g_A : \Omega \setminus \Omega^{(\infty)} \rightarrow C([c_1, \frac{1}{2}])$, defined by

$$g_A(x_1, x_2, \dots)(c) = 2 \sum_{x_j \leq A} x_j^{-2c} + h_A(c),$$

is continuous \mathbf{P} almost everywhere (cf. the proofs of Lemma 2.4 and Lemma 5.1). Hence it follows from [18, Thm. 1'] and [5, Thm. 2.7] that the $C([c_1, \frac{1}{2}])$ -valued random function

$$c \mapsto 2 \sum_{\mathcal{V}_j \leq A} \mathcal{V}_j^{-2c} + h_A(c) = \int_0^1 V^{-2c} dN_n(V) + \int_1^A V^{-2c} dR_n(V)$$

converges in distribution to

$$c \mapsto 2 \sum_{T_j \leq A} T_j^{-2c} + h_A(c) = \int_0^1 V^{-2c} dN(V) + \int_1^A V^{-2c} dR(V)$$

as $n \rightarrow \infty$. The theorem now follows from this fact, Lemma 2.6 and Corollary 6.1 using the Lévy-Prohorov metric (see (5.1)) in a way almost identical to the one in the proof of Theorem 1.1 on p. 24. \square

Proof of Corollary 1.5. Let $\frac{1}{4} < c_1 < c_2 \leq \frac{1}{2}$ be given. To start with, we assume $c_2 < \frac{1}{2}$. Let \mathcal{C} be the following open subset of $C([c_1, c_2])$:

$$\mathcal{C} := \left\{ f \in C([c_1, c_2]) \mid f(c) < 0 \text{ for all } c \in [c_1, c_2] \right\}.$$

Note that

$$\partial\mathcal{C} = \left\{ f \in C([c_1, c_2]) \mid \sup_{c \in [c_1, c_2]} f(c) = 0 \right\}.$$

Let μ_H be the distribution of the $C([c_1, c_2])$ -valued random function $c \mapsto H(c)$. We claim that

$$(6.2) \quad \mu_H(\partial\mathcal{C}) = 0.$$

To prove this, recall that since $0 < T_1 < T_2 < \dots$ are the points of a Poisson process \mathcal{P} on the positive real line with constant intensity $\frac{1}{2}$, they can be realized as the partial sums of an infinite sequence of independent random variables which each has the exponential distribution with parameter $\frac{1}{2}$ (cf. [10, Sec. 4.1]). It follows from this that if we parametrize Ω by the homeomorphism $J : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \Omega \rightarrow \Omega$ given by $J(u, v, \mathbf{z}) = \mathbf{x}$ with $x_1 = u$, $x_2 = u + v$ and $x_j = u + v + z_{j-2}$ for $j \geq 3$, then

$$d\mathbf{P}(\mathbf{x}) = \frac{1}{4} e^{-\frac{1}{2}u} e^{-\frac{1}{2}v} du dv d\mathbf{P}(\mathbf{z}).$$

Hence

$$\mu_H(\partial\mathcal{C}) = \frac{1}{4} \int_{\Omega} \int_0^{\infty} \int_0^{\infty} I(J(u, v, \mathbf{z}) \in \mathcal{S}) e^{-\frac{1}{2}u} e^{-\frac{1}{2}v} du dv d\mathbf{P}(\mathbf{z}),$$

where

$$\mathcal{S} = \left\{ \mathbf{x} \in \Omega \mid \sup_{c \in [c_1, c_2]} H(c) = 0 \right\}.$$

Substituting $v = y - u$ we get

$$(6.3) \quad \mu_H(\partial\mathcal{C}) = \frac{1}{4} \int_{\Omega} \int_0^{\infty} \int_0^y I(J(u, y - u, \mathbf{z}) \in \mathcal{S}) e^{-\frac{1}{2}y} du dy d\mathbf{P}(\mathbf{z}).$$

Now for a given point $\mathbf{x} = J(u, y - u, \mathbf{z})$ we have (assuming $\mathbf{x} \in \Omega_{1/2}$, or equivalently $\mathbf{z} \in \Omega_{1/2}$)

$$(6.4) \quad \begin{aligned} H(c) &= \int_0^{\infty} V^{-2c} dR(V) = \int_0^{x_2} V^{-2c} dR(V) + \int_{x_2}^{\infty} V^{-2c} dR(V) \\ &= 2u^{-2c} + 2y^{-2c} - \frac{y^{1-2c}}{1-2c} + \int_{x_2}^{\infty} V^{-2c} dR(V), \end{aligned}$$

where the last integral is independent of u for given y, \mathbf{z} . Since in fact all terms in the second line of (6.4) except the first are independent of u , and u^{-2c} is a decreasing function of $u > 0$ for every fixed $c \in [c_1, c_2]$, it follows that if $J(u, y - u, \mathbf{z}) \in \mathcal{S}$ for some $0 < u < y$ and $\mathbf{z} \in \Omega_{1/2}$, then $J(u', y - u', \mathbf{z}) \notin \mathcal{S}$ for all u' with $0 < u' < u$ or $u < u' < y$. Hence the innermost integral in (6.3) vanishes for all $y > 0$ and $\mathbf{z} \in \Omega_{1/2}$, and we conclude that (6.2) holds.

Using (6.2), the first part of the corollary now follows from Theorem 1.1 and [5, Thm. 2.1].

In the remaining case $\frac{1}{4} < c_1 < c_2 = \frac{1}{2}$ we consider instead the open set

$$\widehat{\mathcal{C}} := \left\{ f \in C([c_1, \tfrac{1}{2}]) \mid f(c) - \frac{1}{1-2c} < 0 \text{ for all } c \in [c_1, \tfrac{1}{2}) \right\},$$

and let $\mu_{\widehat{H}}$ be the distribution of the $C([c_1, \frac{1}{2}])$ -valued random function $c \mapsto \widehat{H}(c)$. Now

$$\mu_{\widehat{H}}(\partial\widehat{\mathcal{C}}) = 0$$

holds, with almost the same proof as before. (Indeed, this boils down to proving that the triple integral in (6.3) vanishes, where now

$$\mathcal{S} = \left\{ \mathbf{x} \in \Omega \mid \sup_{c \in [c_1, \frac{1}{2})} H(c) = 0 \right\},$$

and the same argument as before applies, since $\lim_{c \rightarrow \frac{1}{2}-} H(c) = -\infty$ for all $\mathbf{x} \in \Omega_{1/2}$.) Hence by Theorem 6.2 and [5, Thm. 2.1] we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{Prob}_{\mu_n} \left\{ L \in X_n \mid \widehat{E}_n(L, cn) - \frac{1}{1-2c} < 0 \text{ for all } c \in [c_1, \tfrac{1}{2}) \right\} \\ = \text{Prob} \left\{ \widehat{H}(c) - \frac{1}{1-2c} < 0 \text{ for all } c \in [c_1, \tfrac{1}{2}) \right\}. \end{aligned}$$

This implies that the first part of the corollary holds also when $c_2 = \frac{1}{2}$.

In order to prove $0 < f(c_1, c_2) < 1$ for general $\frac{1}{4} < c_1 < c_2 \leq \frac{1}{2}$, we let $\Omega(A) = \{\mathbf{x} \in \Omega \mid x_1 > A\}$ for $A > 0$. Clearly for any $\mathbf{x} \in \Omega(A)$ we have

$$\int_0^A V^{-2c} dR(V) = - \int_0^A V^{-2c} dV = - \frac{A^{1-2c}}{1-2c}$$

for all $c \in [c_1, c_2] \setminus \{\frac{1}{2}\}$. Hence, by differentiation with respect to c , we find that for all $A > 1$ and $\mathbf{x} \in \Omega(A)$ we have $\int_0^A V^{-2c} dR(V) \leq -e \log A < 0$ for all $c \in [c_1, c_2] \setminus \{\frac{1}{2}\}$. Recall from Lemma 2.6 that given $\varepsilon > 0$ there exists $A > 1$ such that with probability $\geq 1 - \varepsilon$ we have $\sup_{c \in [c_1, c_2]} \left| \int_A^\infty V^{-2c} dR(V) \right| \leq \varepsilon$. Note that, since the Poisson process \mathcal{P} may be realized as the superposition of a Poisson process on $(0, A)$ and an independent Poisson process on (A, ∞) , both with constant intensity $\frac{1}{2}$ (cf., e.g., [10, Sec. 2.2]), and since furthermore $\int_A^\infty V^{-2c} dR(V)$ only depends on those points of \mathcal{P} which belong to (A, ∞) , the probability of $\sup_{c \in [c_1, c_2]} \left| \int_A^\infty V^{-2c} dR(V) \right| \leq \varepsilon$ remains unchanged if we condition on $\mathbf{x} \in \Omega(A)$. Hence, for small enough ε and large enough A , we have $f(c_1, c_2) \geq (1 - \varepsilon) \mathbf{P}(\Omega(A)) > 0$. Finally, by a similar argument where we instead condition on the event $N(A) = B$ for some large B , we also obtain $f(c_1, c_2) \leq \mathbf{P}\{\mathbf{x} \in \Omega \mid H(c_1) < 0\} < 1$. \square

Theorem 6.2 also has the following corollary.

Corollary 6.3. *The random variable*

$$\widehat{E}_n(\cdot, \tfrac{n}{2}) = \lim_{c \rightarrow \frac{1}{2}} \left(V_n^{-2c} E_n(\cdot, cn) + \frac{1}{1-2c} \right)$$

converges in distribution to Z_0 as $n \rightarrow \infty$.

Proof. Given $c_1 \in (\frac{1}{4}, \frac{1}{2})$ the evaluation map $C([c_1, \frac{1}{2}]) \ni f \mapsto f(\frac{1}{2})$ is continuous. Hence the desired result follows from Theorem 6.2 and [5, Thm. 2.7]. \square

As a consequence of this result we obtain an easy proof of Theorem 1.4.

Proof of Theorem 1.4. First, applying the functional equation (1.1) and (1.2), we get

$$(6.5) \quad E_n(L, \frac{n}{2} - s) = \pi^{\frac{n}{2}-s} \Gamma(\frac{n}{2} - s)^{-1} F_n(L, \frac{n}{2} - s) = \pi^{\frac{n}{2}-s} \Gamma(\frac{n}{2} - s)^{-1} F_n(L^*, s) \\ = \pi^{\frac{n}{2}-2s} \Gamma(\frac{n}{2} - s)^{-1} \Gamma(s) E_n(L^*, s).$$

We are interested in this relation when s is small. Using (1.3) and basic knowledge about the gamma function we have, for s sufficiently small,

$$\pi^{\frac{n}{2}-2s} = \pi^{\frac{n}{2}} \left(1 - 2(\log \pi)s + O(s^2) \right); \\ \Gamma(\frac{n}{2} - s)^{-1} = \left(\Gamma(\frac{n}{2}) - \Gamma'(\frac{n}{2})s + O(s^2) \right)^{-1} \\ = \Gamma(\frac{n}{2})^{-1} \left(1 - \frac{\Gamma'(\frac{n}{2})}{\Gamma(\frac{n}{2})}s + O(s^2) \right)^{-1} = \Gamma(\frac{n}{2})^{-1} \left(1 + \frac{\Gamma'(\frac{n}{2})}{\Gamma(\frac{n}{2})}s + O(s^2) \right); \\ \Gamma(s) = s^{-1} \Gamma(s+1) = s^{-1} - \gamma + O(s); \\ E_n(L^*, s) = - \left(1 - (h_n(L) - 2 \log(2\pi))s + O(s^2) \right),$$

where γ is Euler's constant and the implied constants are allowed to depend on n . Using these expansions in (6.5) yields

$$(6.6) \quad E_n(L, \frac{n}{2} - s) = -\pi^{\frac{n}{2}} \Gamma(\frac{n}{2})^{-1} \left(s^{-1} + \left(2 \log 2 + \frac{\Gamma'(\frac{n}{2})}{\Gamma(\frac{n}{2})} - h_n(L) - \gamma \right) + O(s) \right).$$

Writing $\frac{n}{2} - s$ as cn and using the relation $\pi^{\frac{n}{2}} \Gamma(\frac{n}{2})^{-1} = \frac{1}{2} \omega_n$ we get, for $|c - \frac{1}{2}|$ sufficiently small,

$$E_n(L, cn) = -\frac{\omega_n}{n} \left(\frac{1}{1-2c} + \frac{n}{2} \left(2 \log 2 + \frac{\Gamma'(\frac{n}{2})}{\Gamma(\frac{n}{2})} - h_n(L) - \gamma \right) + O(|c - \frac{1}{2}|) \right).$$

Since we furthermore have

$$V_n^{-2c} = \frac{n}{\omega_n} \left(\frac{\omega_n}{n} \right)^{1-2c} = \frac{n}{\omega_n} \left(1 + (\log \omega_n - \log n)(1-2c) + O(|c - \frac{1}{2}|^2) \right),$$

we obtain

$$V_n^{-2c} E_n(L, cn) \\ = - \left(\frac{1}{1-2c} + \log \omega_n - \log n + \frac{n}{2} \left(2 \log 2 + \frac{\Gamma'(\frac{n}{2})}{\Gamma(\frac{n}{2})} - h_n(L) - \gamma \right) + O(|c - \frac{1}{2}|) \right).$$

Hence, we conclude that

$$(6.7) \quad \lim_{c \rightarrow \frac{1}{2}} \left(V_n^{-2c} E_n(L, cn) + \frac{1}{1-2c} \right) = \log n - \log \omega_n + \frac{n}{2} \left(h_n(L) + \gamma - 2 \log 2 - \frac{\Gamma'(\frac{n}{2})}{\Gamma(\frac{n}{2})} \right).$$

Next we study the asymptotics of (6.7) as $n \rightarrow \infty$. Using (4.8) and Stirling's formula we get

$$\begin{aligned} \lim_{c \rightarrow \frac{1}{2}} \left(V_n^{-2c} E_n(L, cn) + \frac{1}{1-2c} \right) &= \log n - \frac{n}{2} \log \left(\frac{2\pi e}{n} \right) - \frac{1}{2} \log \left(\frac{n}{\pi} \right) \\ &\quad + o(1) + \frac{n}{2} \left(h_n(L) + \gamma - 2 \log 2 - \log \left(\frac{n}{2} \right) + n^{-1} + O(n^{-2}) \right) \\ &= \frac{1}{2} \log(\pi n) + \frac{1}{2} + \frac{n}{2} \left(h_n(L) - (\log(4\pi) - \gamma + 1) \right) + o(1). \end{aligned}$$

Hence we conclude that

$$2\widehat{E}_n(L, \frac{n}{2}) - \log \pi - 1 = n \left(h_n(L) - (\log(4\pi) - \gamma + 1) \right) + \log n + o(1),$$

where $o(1)$ stands for a certain function of n which is independent of L and which tends to 0 as $n \rightarrow \infty$. Using e.g. the Lévy-Prohorov metric on $\mathcal{P}(\mathbb{R})$ (the set of Borel measures on \mathbb{R}), it now follows from Corollary 6.3 that

$$n \left(h_n(L) - (\log(4\pi) - \gamma + 1) \right) + \log n$$

converges in distribution to $2Z_0 - \log \pi - 1$ as $n \rightarrow \infty$, which is the desired result. \square

Remark 6.4. Our proof shows that Theorem 1.4 is really a special case of Theorem 6.2, and we think this nicely illustrates the power of Theorem 6.2. However, it is worth noticing that considerations involving $C([c_1, \frac{1}{2}])$ -valued random functions are not at all essential for the proof of Theorem 1.4: An alternative proof of Theorem 1.4 can be given by working more directly along the lines of Sarnak and Strömbergsson [15, Sec. 6] and applying the $R_n(V)$ -bound in Theorem 1.3 and our Poisson limit result from [18].

To outline this alternative approach, recall from [15, Sec. 4] that

$$(6.8) \quad h_n(L) = \log(4\pi) - \gamma - \frac{2}{n} + \sum_{\mathbf{m} \in L^*}' G(0, \pi |\mathbf{m}|^2) + \sum_{\mathbf{m} \in L}' G\left(\frac{n}{2}, \pi |\mathbf{m}|^2\right),$$

where we call the two sums above $J(L)$ and $H(L)$ respectively. Using the same notation as in Section 4 we have, for any $A > 0$,

$$(6.9) \quad \begin{aligned} H(L) &= \int_0^\infty G\left(\frac{n}{2}, \pi \left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dN_n(V) = \int_0^A G\left(\frac{n}{2}, \pi \left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dN_n(V) \\ &\quad + \int_A^\infty G\left(\frac{n}{2}, \pi \left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dV + \int_A^\infty G\left(\frac{n}{2}, \pi \left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dR_n(V). \end{aligned}$$

The last integral in (6.9) can be bounded using Theorem 1.3 and Lemma 4.5. (The computations are exactly as in the proof of Lemma 4.6 but a tiny bit simpler as we are working only with $c = \frac{1}{2}$, instead of aiming at a uniform bound over the interval $c \in [c_1, \frac{1}{2}]$.) The result is that the random variable

$$n \int_A^\infty G\left(\frac{n}{2}, \pi \left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dR_n(V)$$

converges in distribution to the constant 0, as $A, n \rightarrow \infty$. (Naturally, this also follows as a consequence of Lemma 4.6, since $K_{1/2, n} = \frac{2}{n}$.) Regarding the first integral in the right hand side of (6.9), the same argument as in Proposition 4.3 (cf. also (4.9)

and Lemma 4.2) shows that, for fixed $A > 0$, the distributions of the two random variables

$$n \int_0^A G\left(\frac{n}{2}, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dN_n(V) \quad \text{and} \quad 2 \int_0^A V^{-1} dN_n(V)$$

have Lévy-Prohorov distance tending to 0 as $n \rightarrow \infty$. Furthermore, applying [18, Thm. 1'] and [5, Thm. 2.7] in the usual way (this time for real-valued random variables), it follows that the random variable $2 \int_0^A V^{-1} dN_n(V)$ converges in distribution to $2 \int_0^A V^{-1} dN(V)$ as $n \rightarrow \infty$. Finally, the middle integral in the right hand side of (6.9) can be evaluated asymptotically as $n \rightarrow \infty$, for example as follows. Using Lemma 4.1 and Lemma 4.2 we find that, for fixed $A > 0$ and with an arbitrary fixed constant $0 < \delta < \frac{1}{2e}$,

$$\begin{aligned} \int_A^\infty G\left(\frac{n}{2}, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dV &= \lim_{s \rightarrow \frac{n}{2}-} \left(\frac{1}{\frac{n}{2} - s} - \int_0^A G\left(s, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dV \right) \\ &= \left(\lim_{s \rightarrow \frac{n}{2}-} \frac{1 - \Gamma(\frac{n}{2})^{-1} \Gamma(s) (\pi(\frac{nA}{\omega_n})^{2/n})^{\frac{n}{2}-s}}{\frac{n}{2} - s} \right) + O(e^{-\delta n}) \\ &= \frac{\Gamma'(\frac{n}{2})}{\Gamma(\frac{n}{2})} - \log\left(\pi\left(\frac{nA}{\omega_n}\right)^{2/n}\right) + O(e^{-\delta n}) \\ &= 1 - n^{-1} \log n - n^{-1} (1 + \log \pi + 2 \log A + o(1)) \end{aligned}$$

as $n \rightarrow \infty$. Collecting these results, and also using the fact that the random variable $2 \int_0^A V^{-1} dN(V) - 2 \log A$ converges in distribution to $2Z_0$ as $A \rightarrow \infty$, we conclude that the random variable $n(H(L) - 1) + \log n$ converges in distribution to $2Z_0 - \log \pi - 1$ as $n \rightarrow \infty$.

Similarly,

$$\begin{aligned} J(L^*) &= \int_0^\infty G\left(0, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dN_n(V) \\ &= \int_0^\infty G\left(0, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dV + \int_0^\infty G\left(0, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dR_n(V). \end{aligned}$$

Here the first integral in the right hand side can be evaluated explicitly by Lemma 4.1, and equals $\mathbb{E}(J(L^*)) = \frac{2}{n}$. The second integral in the right hand side can be bounded using Theorem 1.3 and Lemma 4.5 in the same way as in the proof of Proposition 4.7; the result is that the random variable

$$n \int_0^\infty G\left(0, \pi\left(\frac{nV}{\omega_n}\right)^{\frac{2}{n}}\right) dR_n(V)$$

converges in distribution to the constant 0, as $n \rightarrow \infty$. (The same result also follows as a consequence of Proposition 4.7, for $c = \frac{1}{2}$.) Hence we conclude that the random variable $nJ(L^*)$ (and hence also $nJ(L)$) converges in distribution to the constant 2 as $n \rightarrow \infty$. Theorem 1.4 now follows from (6.8) and the above limit results for $n(H(L) - 1) + \log n$ and $nJ(L^*)$. \square

7. PROOF OF THEOREM 1.6

In this section we prove Theorem 1.6. The proof is based on a study of the joint moments of an explicit truncation of

$$\left((2c_1 - \tfrac{1}{2})^{\frac{1}{2}} H(c_1), \dots, (2c_m - \tfrac{1}{2})^{\frac{1}{2}} H(c_m) \right).$$

To be more precise we will, for $\delta > 0$, consider the random vector

$$(7.1) \quad \left((2c_1 - \tfrac{1}{2})^{\frac{1}{2}} H(c_1, \delta), \dots, (2c_m - \tfrac{1}{2})^{\frac{1}{2}} H(c_m, \delta) \right),$$

where

$$H(c, \delta) := \int_{\delta}^{\infty} V^{-2c} dR(V).$$

In order to calculate the joint moments of the random vector (7.1) we first prove a formula closely related to [18, Prop. 3].

Proposition 7.1. *Let $k \geq 1$ and denote by $\mathcal{P}'(k)$ the set of partitions of $\{1, \dots, k\}$ containing no singleton sets. For $1 \leq j \leq k$ let $f_j : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be functions satisfying $\prod_{j \in B} f_j \in L^1(\mathbb{R}_{\geq 0})$ for every nonempty subset $B \subseteq \{1, \dots, k\}$. Then*

$$\mathbb{E} \left(\prod_{j=1}^k \int_0^{\infty} f_j(V) dR(V) \right) = \sum_{P \in \mathcal{P}'(k)} 2^{k-\#P} \prod_{B \in P} \left(\int_0^{\infty} \prod_{j \in B} f_j(V) dV \right).$$

Remark 7.2. In particular, when $1 \leq k \leq 3$ Proposition 7.1 gives

$$\begin{aligned} \mathbb{E} \left(\int_0^{\infty} f_1(V) dR(V) \right) &= 0; \\ \mathbb{E} \left(\prod_{j=1}^2 \int_0^{\infty} f_j(V) dR(V) \right) &= 2 \int_0^{\infty} f_1(V) f_2(V) dV; \\ \mathbb{E} \left(\prod_{j=1}^3 \int_0^{\infty} f_j(V) dR(V) \right) &= 4 \int_0^{\infty} f_1(V) f_2(V) f_3(V) dV. \end{aligned}$$

Proof of Proposition 7.1. Let $K = \{1, \dots, k\}$. Note that for each $1 \leq j \leq k$ we have

$$\int_0^{\infty} f_j(V) dR(V) = 2 \sum_{n=1}^{\infty} f_j(T_n) - \int_0^{\infty} f_j(V) dV.$$

Using this observation together with [18, Prop. 3] we get

$$\begin{aligned} (7.2) \quad & \mathbb{E} \left(\prod_{j=1}^k \int_0^{\infty} f_j(V) dR(V) \right) \\ &= \sum_{A \subset K} (-1)^{\#(K \setminus A)} \left(\prod_{j \in K \setminus A} \int_0^{\infty} f_j(V) dV \right) \mathbb{E} \left(\prod_{j \in A} 2 \sum_{n=1}^{\infty} f_j(T_n) \right) \\ &= \sum_{A \subset K} \sum_{P \in \mathcal{P}(A)} (-1)^{\#(K \setminus A)} 2^{\#A - \#P} \left(\prod_{j \in K \setminus A} \int_0^{\infty} f_j(V) dV \right) \prod_{B \in P} \int_0^{\infty} \prod_{j \in B} f_j(V) dV, \end{aligned}$$

where $\mathcal{P}(A)$ denotes the set of partitions of the set A . Given $A \subset K$ and $P \in \mathcal{P}(A)$ we define $P'(A, P)$ to be the partition

$$P'(A, P) := \{\{j\} \mid j \in K \setminus A\} \cup P$$

of K . Rewriting the right hand side of (7.2) in terms of partitions of K yields

$$(7.3) \quad \sum_{A \subset K} \sum_{P \in \mathcal{P}(A)} (-1)^{\#(K \setminus A)} 2^{\#A - \#P} \prod_{B \in P'(A, P)} \int_0^\infty \prod_{j \in B} f_j(V) dV.$$

For each partition $P' \in \mathcal{P}(K)$ we let $S(P') \subset K$ denote the union of the singleton sets in P' . Note that in the double sum (7.3) we have $P'(A, P) = P'$ for exactly $2^{\#S(P')}$ pairs (A, P) . Indeed, when $K \setminus A$ runs through all subsets of $S(P')$ there exists, for each such A , a unique partition $P \in \mathcal{P}(A)$ with $P'(A, P) = P'$. We conclude that (7.3) equals

$$\begin{aligned} & \sum_{P' \in \mathcal{P}(K)} 2^{k - \#P'} \left(\sum_{C \subset S(P')} (-1)^{\#C} \right) \prod_{B \in P'} \int_0^\infty \prod_{j \in B} f_j(V) dV \\ &= \sum_{\substack{P' \in \mathcal{P}(K) \\ S(P') = \emptyset}} 2^{k - \#P'} \prod_{B \in P'} \int_0^\infty \prod_{j \in B} f_j(V) dV, \end{aligned}$$

which is the desired result. \square

We note that the functions

$$g_{c, \delta}(V) := V^{-2c} I(V \geq \delta)$$

do not satisfy the assumption in Proposition 7.1 for any choice of $c \in (\frac{1}{4}, \frac{1}{2})$ and $\delta > 0$. However, by an approximation argument we get the following corollary.

Corollary 7.3. *Let $k \geq 1$ and let $\delta > 0$ and $\frac{1}{4} < c_1 \leq \dots \leq c_k < \frac{1}{2}$ be fixed. Then*

$$\mathbb{E} \left(\prod_{j=1}^k H(c_j, \delta) \right) = \sum_{P \in \mathcal{P}'(k)} 2^{k - \#P} \delta^{\#P - 2 \sum_{j=1}^k c_j} \prod_{B \in P} \frac{1}{2^{\sum_{j \in B} c_j - 1}}.$$

Proof. For all $\frac{1}{4} < c < \frac{1}{2}$ and $A > \delta$ we let

$$f_A(c, \delta) := \int_\delta^A V^{-2c} dR(V) = 2 \sum_{\delta < T_j \leq A} T_j^{-2c} - \frac{A^{1-2c} - \delta^{1-2c}}{1-2c}.$$

As in Lemma 2.4 we find that $f_A(c, \delta)$ is a measurable function on Ω . We furthermore note that Lemma 2.6 implies that the random vector

$$(f_A(c_1, \delta), \dots, f_A(c_k, \delta))$$

tends in distribution to

$$(H(c_1, \delta), \dots, H(c_k, \delta))$$

as $A \rightarrow \infty$. Hence it follows from [5, Thm. 2.7] that $\prod_{j=1}^k f_A(c_j, \delta)$ converges in distribution to $\prod_{j=1}^k H(c_j, \delta)$ as $A \rightarrow \infty$.

Now, applying Proposition 7.1, we get

$$\mathbb{E} \left(\left| \prod_{j=1}^k f_A(c_j, \delta) \right|^2 \right) = \sum_{P \in \mathcal{P}'(2k)} 2^{2k - \#P} \prod_{B \in P} \int_\delta^A \prod_{j \in B} V^{-2\tilde{c}_j} dV,$$

where $\tilde{c}_{2j} = \tilde{c}_{2j-1} = c_j$ for $1 \leq j \leq k$. By the dominated convergence theorem, together with the fact that $\#B \geq 2$ for all $B \in P \in \mathcal{P}'(2k)$, we have

$$\lim_{A \rightarrow \infty} \mathbb{E} \left(\left| \prod_{j=1}^k f_A(c_j, \delta) \right|^2 \right) = \sum_{P \in \mathcal{P}'(2k)} 2^{2k-\#P} \prod_{B \in P} \int_{\delta}^{\infty} \prod_{j \in B} V^{-2\tilde{c}_j} dV$$

and hence, in particular, it follows that $\sup_{A > \delta} \mathbb{E} \left(\left| \prod_{j=1}^k f_A(c_j, \delta) \right|^2 \right) < \infty$. Similarly we find that

$$\lim_{A \rightarrow \infty} \mathbb{E} \left(\prod_{j=1}^k f_A(c_j, \delta) \right) = \sum_{P \in \mathcal{P}'(k)} 2^{k-\#P} \prod_{B \in P} \int_{\delta}^{\infty} \prod_{j \in B} V^{-2c_j} dV$$

and the corollary now follows from [4, Cor. to Thm. 25.12]. \square

In the special case where $c_1 = \dots = c_k = c$, Corollary 7.3 gives

$$(7.4) \quad \mathbb{E}(H(c, \delta)^k) = \sum_{P \in \mathcal{P}'(k)} 2^{k-\#P} \delta^{\#P-2kc} \prod_{B \in P} \frac{1}{2c\#B-1}.$$

In the next lemma we will consider the rescaled variable

$$(7.5) \quad \mathcal{H}(c, \delta) := (2c - \tfrac{1}{2})^{\frac{1}{2}} \delta^{2c-\frac{1}{2}} H(c, \delta),$$

which by (7.4) satisfies $\mathbb{E}(\mathcal{H}(c, \delta)) = 0$, $\mathbb{E}(\mathcal{H}(c, \delta)^2) = 1$ and

$$(7.6) \quad \mathbb{E}(\mathcal{H}(c, \delta)^k) = (2c - \tfrac{1}{2})^{\frac{k}{2}} \sum_{P \in \mathcal{P}'(k)} 2^{k-\#P} \delta^{\#P-\frac{k}{2}} \prod_{B \in P} \frac{1}{2c\#B-1}, \quad k \geq 3.$$

If $k \geq 3$ is odd, then for every $P \in \mathcal{P}'(k)$ we have $\#\{B \in P \mid \#B = 2\} \leq \frac{1}{2}(k-3)$. Hence it follows from (7.6) that, for fixed $\delta > 0$ and odd $k \geq 3$, we have $\lim_{c \rightarrow \frac{1}{4}+} \mathbb{E}(\mathcal{H}(c, \delta)^k) = 0$. Similarly we find that, for fixed $\delta > 0$ and even $k \geq 4$, we have

$$\begin{aligned} \lim_{c \rightarrow \frac{1}{4}+} \mathbb{E}(\mathcal{H}(c, \delta)^k) &= \#\{P \in \mathcal{P}'(k) \mid \#B = 2, \forall B \in P\} \\ &= \binom{k}{2, \dots, 2} \frac{1}{(k/2)!} = (k-1)!! \end{aligned}$$

Since these limits coincide with the corresponding moments of the distribution $N(0, 1)$ and normal distributions are determined by their moments, we conclude that, for any fixed $\delta > 0$, $\mathcal{H}(c, \delta)$ converges in distribution to $N(0, 1)$ as $c \rightarrow \frac{1}{4}+$. More generally, we have the following result.

Lemma 7.4. *Fix $m \in \mathbb{Z}_{\geq 1}$ and let $c_j = \frac{1}{4} + \eta_j$ with $\eta_j \in (0, \frac{1}{4})$ for $1 \leq j \leq m$. If $\delta > 0$ is fixed and (η_1, \dots, η_m) tends to the zero vector in \mathbb{R}^m in such a way that $\eta_j/\eta_{j+1} \rightarrow 0$ for each $1 \leq j \leq m-1$, then the m -dimensional random vector $(\mathcal{H}(c_1, \delta), \dots, \mathcal{H}(c_m, \delta))$ converges in distribution to the distribution of m independent $N(0, 1)$ -variables.*

Proof. It remains to consider the case where $m \geq 2$. Let $k_1, \dots, k_m \in \mathbb{Z}_{\geq 0}$ satisfying $k = k_1 + \dots + k_m \geq 1$ be given and let

$$\tilde{c}_j = \begin{cases} c_1 & \text{if } 1 \leq j \leq k_1, \\ c_2 & \text{if } k_1 < j \leq k_1 + k_2, \\ \vdots & \\ c_m & \text{if } k_1 + \dots + k_{m-1} < j \leq k. \end{cases}$$

It follows from Corollary 7.3 and (7.5) that

$$(7.7) \quad \mathbb{E} \left(\prod_{j=1}^m \mathcal{H}(c_j, \delta)^{k_j} \right) = \left(\prod_{j=1}^m (2\eta_j)^{\frac{k_j}{2}} \right) \delta^{2 \sum_{j=1}^m k_j \eta_j} \sum_{P \in \mathcal{P}'(k)} 2^{k - \#P} \delta^{\#P - 2 \sum_{j=1}^m k_j c_j} \prod_{B \in P} \frac{1}{2^{\sum_{j \in B} \tilde{c}_j - 1}}.$$

In this sum, the contribution from a given partition $P \in \mathcal{P}'(k)$ is, in the limit under consideration, writing M_{i_1, i_2} for the number of elements $B \in P$ which satisfy $\#B = 2$, $\min B \in (k_1 + \dots + k_{i_1-1}, k_1 + \dots + k_{i_1}]$ and $\max B \in (k_1 + \dots + k_{i_2-1}, k_1 + \dots + k_{i_2}]$ (here $k_1 + \dots + k_0 := 0$),

$$(7.8) \quad \begin{aligned} & \asymp \prod_{j=1}^m \eta_j^{\frac{k_j}{2}} \prod_{\substack{B \in P \\ \#B=2}} \frac{1}{2^{\sum_{j \in B} \tilde{c}_j - 1}} \\ & \asymp \prod_{j=1}^m \eta_j^{\frac{k_j}{2}} \prod_{1 \leq i_1 \leq i_2 \leq m} (\eta_{i_1} + \eta_{i_2})^{-M_{i_1, i_2}} \asymp \prod_{j=1}^m \eta_j^{\frac{k_j}{2}} \prod_{1 \leq i_1 \leq i_2 \leq m} \eta_{i_2}^{-M_{i_1, i_2}} \\ & = \left(\prod_{j=1}^{m-1} \left(\frac{\eta_j}{\eta_{j+1}} \right)^{\frac{1}{2} \sum_{\ell=1}^j k_\ell - \sum_{1 \leq i_1 \leq i_2 \leq j} M_{i_1, i_2}} \right) \eta_m^{\frac{1}{2} \sum_{\ell=1}^m k_\ell - \sum_{1 \leq i_1 \leq i_2 \leq m} M_{i_1, i_2}}. \end{aligned}$$

Hence, since by definition we have $\sum_{1 \leq i_1 \leq i_2 \leq j} M_{i_1, i_2} \leq \frac{1}{2} \sum_{\ell=1}^j k_\ell$ for each $1 \leq j \leq m$, the expression in (7.8) tends to zero unless

$$(7.9) \quad \sum_{1 \leq i_1 \leq i_2 \leq j} M_{i_1, i_2} = \frac{1}{2} \sum_{\ell=1}^j k_\ell, \quad \forall j \in \{1, \dots, m\}.$$

Now suppose that $P \in \mathcal{P}'(k)$ gives a non-zero limit contribution to (7.7). From (7.9) we get $M_{1,1} = \frac{1}{2}k_1$, which implies that $M_{1,j} = 0$ for all $2 \leq j \leq m$. Next (7.9) gives $M_{1,1} + M_{1,2} + M_{2,2} = \frac{1}{2}(k_1 + k_2)$. By our previous observations we must have $M_{2,2} = \frac{1}{2}k_2$ and hence it follows that $M_{2,j} = 0$ for all $3 \leq j \leq m$. Continuing in the same way we find that (7.9) forces $M_{j,j} = \frac{1}{2}k_j$ for all $1 \leq j \leq m$ and $M_{i,j} = 0$ whenever $i < j$. Conversely, we note that these conditions imply that (7.9) holds. Thus, in particular, the moment in (7.7) tends to zero unless all k_j are even. Furthermore, for each partition $P \in \mathcal{P}'(k)$ satisfying (7.9) the contribution to (7.7) equals

$$\left(\prod_{j=1}^m (2\eta_j)^{\frac{k_j}{2}} \right) 2^{k - \#P} \delta^{\#P - 2 \sum_{j=1}^m k_j (c_j - \eta_j)} \prod_{B \in P} \frac{1}{2^{\sum_{j \in B} \tilde{c}_j - 1}} = 1.$$

Hence, in the limit under consideration, the moment $\mathbb{E}(\prod_{j=1}^m \mathcal{H}(c_j, \delta)^{k_j})$ tends to the number of partitions $P \in \mathcal{P}'(k)$ satisfying condition (7.9). Recalling the discussion below (7.6) we conclude that

$$\mathbb{E}\left(\prod_{j=1}^m \mathcal{H}(c_j, \delta)^{k_j}\right) \rightarrow \prod_{j=1}^m M_{k_j},$$

where

$$M_k := \begin{cases} 0 & \text{if } k \text{ is odd,} \\ (k-1)!! & \text{if } k \text{ is even.} \end{cases}$$

The lemma follows since a random vector whose coordinates in the standard basis are independent $N(0, 1)$ -variables is determined by its joint moments. \square

Proof of Theorem 1.6. Let $c_j = \frac{1}{4} + \eta_j$ with $\eta_j \in (0, \frac{1}{4})$ for $1 \leq j \leq m$. For convenience of notation we set $\mathcal{H}(c) := (2c - \frac{1}{2})^{\frac{1}{2}} H(c)$. We note that if $\delta > 0$ is fixed and $\mathbf{x} \in \Omega$ is such that $N(\delta) = 0$, then

(7.10)

$$\mathcal{H}(c) = (2c - \frac{1}{2})^{\frac{1}{2}} \left(H(c, \delta) - \int_0^\delta V^{-2c} dV \right) = \delta^{\frac{1}{2}-2c} \mathcal{H}(c, \delta) - (2c - \frac{1}{2})^{\frac{1}{2}} \frac{\delta^{1-2c}}{1-2c},$$

where $\lim_{c \rightarrow \frac{1}{4}+} \delta^{\frac{1}{2}-2c} = 1$ and $\lim_{c \rightarrow \frac{1}{4}+} (2c - \frac{1}{2})^{\frac{1}{2}} \frac{\delta^{1-2c}}{1-2c} = 0$.

Now let $\varepsilon > 0$ be given. Fix $\delta > 0$ small enough to ensure that $\mathbf{P}(N(\delta) = 0) > 1 - \varepsilon$. By Lemma 7.4 there exist numbers $0 < \tilde{\eta}_1 < \dots < \tilde{\eta}_m < \frac{1}{4}$ such that for all vectors (η_1, \dots, η_m) satisfying $0 < \eta_j \leq \tilde{\eta}_j$ ($1 \leq j \leq m$) as well as $0 < \frac{\eta_j}{\eta_{j+1}} < \frac{\tilde{\eta}_j}{\tilde{\eta}_{j+1}}$ ($1 \leq j \leq m-1$), the distribution of $(\mathcal{H}(c_1, \delta), \dots, \mathcal{H}(c_m, \delta))$ is within ε of the distribution of m independent $N(0, 1)$ -variables in the Lévy-Prohorov metric. Furthermore it follows from (7.10) that we can, by possibly shrinking the numbers $\tilde{\eta}_j$, guarantee that

$$\mathbf{P}\left(\left|(\mathcal{H}(c_1), \dots, \mathcal{H}(c_m)) - (\mathcal{H}(c_1, \delta), \dots, \mathcal{H}(c_m, \delta))\right| < \varepsilon\right) > 1 - 2\varepsilon,$$

for all admissible (η_1, \dots, η_m) . The observations above together imply that the distribution of $(\mathcal{H}(c_1), \dots, \mathcal{H}(c_m))$ is, for all admissible (η_1, \dots, η_m) , within 3ε of the distribution of m independent $N(0, 1)$ -variables in the Lévy-Prohorov metric. This concludes the proof. \square

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